

ANALYTIC FAMILIES OF QUANTUM HYPERBOLIC INVARIANTS AND THEIR ASYMPTOTICAL BEHAVIOUR, I

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ABSTRACT. This is the first of two papers in which we organize the quantum hyperbolic invariants of 3-manifolds into families of rational functions defined on certain moduli spaces of geometric structures refining the character varieties, and we analyze the asymptotic behaviour of these families of functions when the “level” $N \rightarrow +\infty$. In this paper we construct new quantum hyperbolic invariants (and the related families of functions) supported by one-cusped hyperbolic 3-manifolds M . As a result, we get for every such an M a family of regular rational functions $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$, indexed by the odd integers $N \geq 3$ and depending on a finite set of (topological) cohomological data (h_f, h_c, k_c) called *weights*, where each function is defined on a determined Abelian covering of degree N^2 of a Zariski open subset, canonically associated to M , of the geometric component of the variety of augmented $PSL(2, \mathbb{C})$ -characters of M . The functions $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$ are computed by state sums refining those previously constructed in [2], where branchings on triangulations are relaxed to a notion of *weak branching* existing for any ideal triangulation, which is required to deal with arbitrary cusped manifolds. We provide also a closed intrinsic formula in terms of weights of the so called “symmetrization factor” that one can extract from any state sum computing $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$, and this eventually leads to a factorization of the quantum invariants themselves. We will face the asymptotic analysis in the second paper.

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1. INTRODUCTION

This is the first of two papers in which we will achieve the following goals:

QHI of cusped manifolds: To extend the quantum hyperbolic invariants (QHI) of a one-cusped hyperbolic 3-manifold M , defined in [2], to functions defined for any $PSL(2, \mathbb{C})$ -character of M in a determined Zariski open subset of the geometric component of its characters variety, and for arbitrary weights. To do so, it will be necessary to introduce (invariant) enhanced state sums over triangulations with “relaxed” structures, so that too demanding branchings are replaced by weak branchings.

Analytic families of QHI: To show that the QHI of *all* the patterns having a same topological support Y (and hence encoding geometric structures on Y , see Section 1.1 and 1.2) fit into a richly structured sequence $\{\mathcal{A}_N(Y)\}$ of families of complex analytic spaces and maps, indexed by the odd integers $N \geq 3$, for which we will provide concrete models. For patterns based on a cusped manifold M , these analytic families have a strong intrinsic meaning related to the $PSL(2, \mathbb{C})$ version of the A -polynomial of M , and also a clean natural relation with the Chern-Simons theory of M .

QH asymptotic problem: Given a sequence $\{\mathcal{P}_N\}$ of patterns having a same topological support Y , and the corresponding sequence $\{\mathcal{H}_N(\mathcal{P}_N)\}$ of QHI, let

$$\mathcal{H}_\infty(\{\mathcal{P}_N\}) := \limsup_{N \rightarrow \infty} (\log |\mathcal{H}_N(\mathcal{P}_N)|/N) .$$

In many cases, in fact in the most natural ones (for instance when $\mathcal{P}_N = \mathcal{P}$ is a constant sequence), it is not hard to check that $\mathcal{H}_\infty(\{\mathcal{P}_N\}) < \infty$. So we can consider the following questions.

- (1) Which are the relations between the geometric structures carried by the sequence $\{\mathcal{P}_N\}$ and the growth rate of the sequence $\{|\mathcal{H}_N(\mathcal{P}_N)|\}$? When is it true that $\mathcal{H}_\infty(\{\mathcal{P}_N\}) \neq 0$?
- (2) Which classical invariants of $\{\mathcal{P}_N\}$ are recovered from $\mathcal{H}_\infty(\{\mathcal{P}_N\})$, when non zero ? What is the geometric content of the asymptotic expansion of $\{\mathcal{H}_N(\mathcal{P}_N)\}$ as $N \rightarrow \infty$?

We will explain, for instance, how the celebrated Kashaev-Murakami-Murakami *Volume Conjecture* is a particular case of such questions, corresponding to constant sequences of patterns associated to links in S^3 . The main idea of our approach is to study, for every topological support Y , the global asymptotic behaviour of the analytic family $\{\mathcal{A}_N(Y)\}$, instead of $\mathcal{H}_\infty(\{\mathcal{P}_N\})$ for a single sequence $\{\mathcal{P}_N\}$ of patterns with topological support Y . We will develop some general tools, like an integral formula of $\mathcal{H}_N(\mathcal{P}_N)$ and an asymptotic equivalent thereof. In the case of one-cusped manifolds, we will be able to treat completely a few interesting examples, that support a conjecture answering the exponential growth rate question in (2). Some intriguing phenomena show up, like the subtle role of the weights that enter the definition of the patterns.

In this first paper we face the first two goals. By the way we point out also a new *factorization* result that eventually holds for all the quantum hyperbolic invariants. In the second one, in collaboration with C. Frohman, we will develop our approach to the asymptotic problem. Let us describe with more details the content of the present paper.

1.1. QHI of patterns based on cusped manifolds. For the aims of this paper, we stipulate that a *cusped manifold* M is an oriented, connected, non-compact complete hyperbolic 3-manifold of finite volume with *one* cusp. Hence M is diffeomorphic to the interior of a compact 3-manifold denoted by V , with boundary ∂V made by one torus component.

A *pattern* $\mathcal{P} = (Y_{\mathcal{P}}, \rho, (h, k))$ based on M consists of a *topological support* $Y_{\mathcal{P}}$, together with additional geometric structures determined by the couple $(\rho, (h, k))$. The topological support has the form

$$Y_{\mathcal{P}} = (V, (h_c, k_c))$$

where

$$(h_c, k_c) \in H^1(V; \mathbb{Z}/2\mathbb{Z}) \times H^1(\partial V; \mathbb{Z})$$

is a so-called *c-weight*, defined by the condition that the “bulk *c-weight*” h_c and the “boundary *c-weight*” k_c satisfy the identity

$$(1) \quad r(k_c) = i^*(h_c)$$

where $r : H^1(\partial V; \mathbb{Z}) \rightarrow H^1(\partial V; \mathbb{Z}/2\mathbb{Z})$ is the coefficient reduction mod(2), and $i^* : H^1(V; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(\partial V; \mathbb{Z}/2\mathbb{Z})$ is induced by the inclusion map $i : \partial V \rightarrow V$.

The pattern \mathcal{P} is obtained by completing $Y_{\mathcal{P}}$ with a couple

$$(\rho, (h_f, k_f))$$

where ρ is a conjugacy class of representations of the fundamental group $\pi_1(V)$ in $PSL(2, \mathbb{C})$ (shortly, ρ is a $PSL(2, \mathbb{C})$ -character of V), and

$$(h_f, k_f) \in H^1(V; \mathbb{Z}/2\mathbb{Z}) \times H^1(\partial V; \mathbb{C})$$

is a so-called *f-weight* (relative to ρ), defined by the condition that the “bulk *f-weight*” h_f and the “boundary *f-weight*” k_f satisfy the following constraint. Up to conjugacy the restriction of ρ to the torus ∂V is valued in the group of complex affine transformations of the plane; hence the linear part (i.e. the “derivative”) of such a restriction defines a class in $H^1(\partial V; \mathbb{C}^*)$. Define $d \in H^1(\partial V; \mathbb{C}/2i\pi\mathbb{Z})$ as the log of this class, with imaginary part in $]-\pi, \pi]$. Then, we require that for all $a \in H_1(\partial V; \mathbb{Z})$,

$$(2) \quad k_f(a) = d(a) \bmod(i\pi); \quad (k_f(a) - d(a))/i\pi = i^*(h_f)(a) \bmod(2) .$$

Finally, collecting the bulk and boundary weights we will often write \mathcal{P} as $(V, \rho, (h, k))$, where

$$(h, k) = ((h_f, h_c), (k_f, k_c)) .$$

In [2, 3], for any cusped manifold M we have defined quantum hyperbolic invariants

$$(3) \quad \mathcal{H}_N(M) := \mathcal{H}_N(V, \rho_{hyp}, (0, 0))$$

that is, for the specific pattern $\mathcal{P} = (V, \rho_{hyp}, (0, 0))$ where ρ_{hyp} is the hyperbolic holonomy of M , and all weights vanish. The invariants $\mathcal{H}_N(M)$ are complex numbers, defined for every odd $N \geq 3$ up to multiplication by a $2N$ th root of unity. In this paper we get the following extension to more general patterns based on M (all terms are defined in Section 4).

Notation. For every $m \in \mathbb{N}$ we write “ $a \equiv_m b$ ” to mean “equal up to multiplication by a power of $\exp(2i\pi/m)$ ”.

Theorem 1.1. *Let M be an arbitrary cusped manifold, $X(M)$ the augmented $PSL(2, \mathbb{C})$ -character variety of M , and $X_0(M)$ the irreducible component of $X(M)$ that contains the hyperbolic holonomy ρ_{hyp} . Then:*

(1) *There is a determined non empty Zariski open subset $\Omega(M)$ of $X_0(M)$ containing ρ_{hyp} , and for every odd $N \geq 3$, enhanced quantum hyperbolic invariants $\mathcal{H}_N^e(V, \rho, (h, k))$ of patterns $(V, \rho, (h, k))$ based on M , defined for every point ρ in $\Omega(M)$ and every weight (h, k) .*

(2) *The invariants $\mathcal{H}_N^e(V, \rho, (h, k))$ are complex numbers defined up to multiplication by a $4N$ th root of unity. Up to this phase ambiguity, when $\rho = \rho_{hyp}$ and the weights (h, k) vanish, they coincide with the quantum hyperbolic invariants $\mathcal{H}_N(M)$.*

(3) *For every odd $N \geq 3$, every topological support $(V, (h_c, k_c))$ based on M , and every class $h_f \in H^1(V; \mathbb{Z}/2\mathbb{Z})$, the invariants $\mathcal{H}_N^e(V, \rho, (h, k))$ define for varying $\rho \in \Omega(M)$ and boundary *f-weight* k_f satisfying (2), a regular rational function*

$$(\mathcal{H}_N^e)^{h_f, h_c, k_c} : \tilde{\Omega}(M)_N \rightarrow \mathbb{C}/\mu_{4N}$$

where $\tilde{\Omega}(M)_N$ is a determined Abelian covering of $\Omega(M)$ of degree N^2 , and μ_{4N} is the group of $4N$ th roots of unity, acting on \mathbb{C} by multiplication.

(4) *Let \mathcal{T} be a QH triangulation of the pattern $(V, \rho, (h, k))$, and $\mathcal{H}_N^e(\mathcal{T})$ the state sum formula of $\mathcal{H}_N^e(V, \rho, (h, k))$ over \mathcal{T} . Denote by $\alpha_N(\mathcal{T})$ the product of the matrix dilogarithm symmetrization factors occurring in the state sum $\mathcal{H}_N^e(\mathcal{T})$, and by $(\mathcal{H}_N^e)^R(\mathcal{T})$ the reduced state sum defined by writing $\mathcal{H}_N^e(\mathcal{T})$ as a product $\alpha_N(\mathcal{T})(\mathcal{H}_N^e)^R(\mathcal{T})$. Then, we get a factorization into invariants*

$$\mathcal{H}_N^e(V, \rho, (h, k)) = \alpha_N(V, \rho, (h, k))(\mathcal{H}_N^e)^R(V, \rho, (h, k))$$

where $\alpha_N(V, \rho, (h, k))$ is defined up to sign, and $(\mathcal{H}_N^e)^R(V, \rho, (h, k))$ is defined up to multiplication by $4N$ th roots of unity. More precisely, denote by

$$\langle\langle \ , \ \rangle\rangle : H^1(\partial V; \mathbb{C}) \times H^1(\partial V; \mathbb{C}) \rightarrow \mathbb{C}$$

the intersection product (i.e. cup product followed by evaluation on the fundamental class), and $\langle\langle \ , \ \rangle\rangle_2$ the intersection product on $H^1(\partial V; \mathbb{Z}/2\mathbb{Z}) \times H^1(\partial V; \mathbb{Z}/2\mathbb{Z})$. Then

$$(4) \quad \alpha_N(\mathcal{T}) \equiv_2 \exp \left(\frac{N-1}{4} \left(\frac{1}{N} \langle\langle k_c, k_f \rangle\rangle + i\pi \langle\langle r(k_c), i^*(h_f) \rangle\rangle_2 \right) \right).$$

Comments:

a) For clarity let us recall that $X(M)$ is a complex algebraic variety and ρ_{hyp} is a regular point of $X(M)$. Hence there is a unique irreducible component $X_0(M)$ of $X(M)$ that contains the hyperbolic holonomy. As M has only one cusp, $X_0(M)$ is an algebraic curve. Hence the Zariski open set $\Omega(M)$ is the complement of a finite set of points in $X_0(M)$.

b) The matrix dilogarithm symmetrization factors, whose product defines $\alpha_N(\mathcal{T})$ in Theorem 1.1 (4), have been introduced in [2] to solve a “symmetrization problem” that implies a full invariance of state sums under enhanced versions of triangulation moves called *transits* (see Section 6). The fact that $\alpha_N(V, \rho, (h, k)) \equiv_2 \alpha_N(\mathcal{T})$ is a well defined invariant follows from the intrinsic formula (4). On the other hand, we *deduce* that $(\mathcal{H}_N^e)^R(V, \rho, (h, k)) \equiv_{4N} (\mathcal{H}_N^e)^R(\mathcal{T})$ is a well defined invariant from the fact that we *already* know this for $\alpha_N(V, \rho, (h, k))$, and for $\mathcal{H}_N^e(V, \rho, (h, k)) \equiv_{4N} \mathcal{H}_N^e(\mathcal{T})$ via a proof based on transit arguments. We stress that we found no other, more direct way to achieve this result. The key reason is that the state sums $(\mathcal{H}_N^e)^R(\mathcal{T})$ have a non local behaviour under branching changes, whereas the transit arguments are local by nature. If M is very gentle, as in Remark 1.8 below, so that the (non enhanced) invariants $\mathcal{H}_N(V, \rho, (h, k))$ are defined up to multiplication by $2N$ th roots of unity, the reduced invariants $(\mathcal{H}_N)^R(V, \rho, (h, k))$ are defined up to multiplication by a $2N$ th root of unity.

c) Point (3) of Theorem 1.1 gives a rough qualitative formulation of the N th analytic family $\mathcal{A}_N(Y)$ associated to the topological support $Y = (V, (h_c, k_c))$ based on M . For each choice of c -weight (h_c, k_c) , the function $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$ itself is an invariant of (M, h_f) . It defines a rational function on a covering space $\tilde{X}_0(M)_N$ of $X_0(M)$ containing $\tilde{\Omega}(M)_N$. The function field $\mathbb{C}(\tilde{X}_0(M)_N)$ is a finite extension of $\mathbb{C}(X_0(M))$, which in turn is generated by the augmented characters of a pair of meridian and longitude curves under the birational isomorphism of $X_0(M)$ onto components of the A -polynomial curve of M (see Section 4). So a natural problem that deserves future investigation is:

Problem 1.2. Find an expression/characterization of the rational functions $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$ in terms of augmented characters of M , or as meromorphic functions on the smooth projective model of $\tilde{X}_0(M)_N$.

For instance, a property that is obvious from the state sum formulas is that the poles of $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$ cover ideal points of $X_0(M)$.

On another hand, by a theorem of Bullock [10], the ring of $SL(2, \mathbb{C})$ -characters $\mathbb{C}[X(M)]$ is isomorphic to $K_{-1}(M)/\sqrt{0}$, where $K_{-1}(M)$ is the Kauffman bracket skein module of M specialized at $A = -1$, with its natural algebra structure, and $\sqrt{0}$ its nilradical.

Problem 1.3. Find skein theoretic constructions of $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$.

By varying the boundary c -weight k_c one obtains a set of functions $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$ which has a very rich structure. Its classical analog is the Chern-Simons complex line, spanned by a single analytic function $\mathcal{H}_0^{h_f}$ defined over a $\mathbb{Z} \times \mathbb{Z}$ -covering space of $\Omega(M)$ (see Section 4.4); $\mathcal{H}_0^{h_f}$ descends to a parallel section of a flat (trivial) \mathbb{C}^* -bundle over $\Omega(M)$ under the action of $\mathbb{Z} \times \mathbb{Z}$ by deck transformations. It is natural to expect a refinement of this action in the quantum world. Because of the $4N$ th roots of unity ambiguities, let us consider the complex vector space \mathbb{L}_N spanned by the $4N$ th powers $((\mathcal{H}_N^e)^{h_f, h_c, k_c})^{4N}$ as k_c varies. One can define a representation of $\mathbb{Z} \times \mathbb{Z} \cong H^1(\partial V; \mathbb{Z})$ on \mathbb{L}_N in the following way: associate to a class $\gamma' \in H^1(\partial V; \mathbb{Z})$ the linear operator $\hat{\gamma}'$ on \mathbb{L}_N defined on the natural basis by

$$(5) \quad \hat{\gamma}' \cdot ((\mathcal{H}_N^e)^{h_f, h_c, k_c})^{4N} = ((\mathcal{H}_N^e)^{h_f, h_c, k_c + \gamma'})^{4N}.$$

It is a consequence of Theorem 1.1 (4) that the subgroup $N\mathbb{Z} \times N\mathbb{Z} \cong H^1(\partial V; N\mathbb{Z})$ acts by scalar multiples of the identity endomorphism of \mathbb{L}_N (the scalars being given by the action on the functions on $\tilde{\Omega}(M)_N$ associated to the invariants $\alpha_N(V, \rho, (h, k))^{4N}$), and that in general a class γ' acts on the function $((\mathcal{H}_N^e)^R)^{h_f, h_c, k_c}^{4N}$ associated to the reduced invariants $(\mathcal{H}_N^e)^R(V, \rho, (h, k))^{4N}$ by a deck transformation on $\tilde{\Omega}(M)_N$. We expect positive answers to the following questions.

Question 1.4. 1) Can one lift the functions $(\mathcal{H}_N^e)^{h_f, h_c, k_c} : \tilde{\Omega}(M)_N \rightarrow \mathbb{C}/\mu_{4N}$ to complex valued functions, possibly up to the cost of fixing some additional structures on M ? 2) Assume that there exist complex valued functions refining $(\mathcal{H}_N^e)^{h_f, h_c, k_c}$ or $((\mathcal{H}_N^e)^R)^{h_f, h_c, k_c}$ as in 1). Denote by $\tilde{\mathbb{L}}_N$ the complex vector space they span. Does there exist an action on $\tilde{\mathbb{L}}_N$ of the quantum torus generated by operators \hat{l} and \hat{m} satisfying $\hat{l}\hat{m} = e^{2i\pi/N}\hat{m}\hat{l}$, which refines the action (5) of $\mathbb{Z} \times \mathbb{Z}$ on \mathbb{L}_N ?

1.2. QHI of other patterns. In order to explain the nature of the enhancement in point (1) of Theorem 1.1, and why it is not straightforward, it is useful to recall a few general facts about the construction of quantum hyperbolic invariants.

QHFT partition functions. Quantum hyperbolic invariants $\mathcal{H}_N(\mathcal{P})$ have been defined in [1, 2, 3] also for a variety of patterns \mathcal{P} not based on cusped manifolds. As above they form sequences of complex numbers, defined for every odd $N \geq 3$ up to multiplication by a $2N$ th root of unity. These QHI are the partition functions of a family of finite dimensional $(2+1)$ -quantum hyperbolic field theories (QHFT) defined in [3]. For such patterns the topological support has the form

$$Y_{\mathcal{P}} = (V, L, (h_c, k_c))$$

where:

- V is a compact oriented connected 3-manifold with (possibly empty) boundary ∂V made by torus components; if $\partial V = \emptyset$ we will often use the notation W instead of V .
- L is a *non-empty* link in the interior of V (the non-empty link is the characterizing property of these patterns);
- bulk & boundary c -weights (h_c, k_c) are formally defined as in (1) for cusped manifolds, by allowing now an arbitrary number of boundary components of V .

The QHFT patterns are obtained by completing $Y_{\mathcal{P}}$ with a couple $(\rho, (h_f, k_f))$, where ρ is a $PSL(2, \mathbb{C})$ -character of V and (h_f, k_f) are bulk and boundary f -weights with respect to ρ , satisfying (2). Often we will write $\mathcal{H}_N(V, L, \rho, (h, k))$ instead of $\mathcal{H}_N(\mathcal{P})$.

When $V = W$ is compact closed, then ρ is a $PSL(2, \mathbb{C})$ -character of W , the boundary weights $k = (k_c, k_f)$ disappear, and $h = (h_c, h_f)$ belongs to $H^1(W; \mathbb{Z}/2\mathbb{Z})^2$. In [1, 2] we constructed the quantum hyperbolic invariants in this particular situation, that is

$$\mathcal{H}_N(W, L, \rho, h)$$

for *every* character ρ and *every* weight h . The first quantum hyperbolic invariants

$$H_N(W, L, \rho) := \mathcal{H}_N(W, L, \rho, 0)$$

were defined in [1]. A further specialization is

$$H_N(S^3, L) := H_N(S^3, L, \rho_{\text{triv}})$$

where ρ_{triv} is the trivial character of S^3 . The following result establishes an important connection between quantum hyperbolic invariants and Jones polynomials.

Theorem 1.5. ([4]) *For every link $L \in S^3$ and every odd $N \geq 3$ one has*

$$\begin{aligned} H_N(S^3, L) &\equiv_{2N} \langle L \rangle_N \\ &= J_N(L)(e^{2i\pi/N}) \end{aligned}$$

where $\langle L \rangle_N$ is the link invariant defined by the enhanced Yang-Baxter operator extending the Kasheev R -matrix, and $J_N(L)(q) \in \mathbb{Z}[q^{\pm 1}]$ is the colored Jones polynomial normalized by $J_N(K_U)(q) = 1$ on the unknot K_U .

Remark 1.6. The second equality in Theorem 1.5 is due to [29]. In [1, 2, 3] we quoted occasionally the first equality as a motivating fact. Later we realized that we were unable to derive a complete proof by starting from the existing literature (in particular [22, 23]). Hence we preferred to provide an independent proof in [4] (see also Remarks 2.9 and 5.4).

When ∂V is non-empty, the construction of QHFT partition functions is more elaborated. In particular, it is necessary that the link L contains (up to isotopy) an essential simple closed curve on each boundary component, which actually encodes a Dehn filling instruction (see [3]). Anyway, also in this case the invariants $\mathcal{H}_N(V, L, \rho, (h, k))$ are defined for *arbitrary* characters and weights.

Relation with the QHI of cusped cusped manifolds. In a sense the patterns based on cusped manifolds are complementary to the QHFT patterns, as for the former the link L is actually *empty*. The proof that the invariants $\mathcal{H}_N(M) = \mathcal{H}_N(V, \rho_{\text{hyp}}, (0, 0))$ (see (3)) are well defined differs to many extents from the one for the QHFT partition functions. It uses the so called “volume rigidity” for cusped manifolds (see eg. [19]), Thurston’s hyperbolic Dehn filling theorem, a construction of certain auxiliary invariants $\mathcal{H}_N(V, \rho_{\text{hyp}}, (0, 0), a)$ that depend a priori on an additional datum “ a ”, and finally a surgery formula. By combining all these results we proved:

Theorem 1.7. ([3], Section 6.2) *Let W_n be a sequence of closed hyperbolic Dehn fillings of M that converges geometrically to M as $n \rightarrow +\infty$. Denote by L_n the geodesic core of the solid torus that fills V to produce W_n , and by ρ_n the hyperbolic holonomy of W_n . For every odd $N \geq 3$ and every additional datum “ a ”, we have*

$$(6) \quad \lim_{n \rightarrow \infty} \mathcal{H}_N(W_n, L_n, \rho_n) = \mathcal{H}_N(V, \rho_{\text{hyp}}, (0, 0), a) .$$

Hence the datum “ a ” is eventually immaterial, and the limit defines an invariant of M , that we denote by $\mathcal{H}_N(M)$.

The normalization $h = (0, 0), k = (0, 0)$ of the weights that appears on the right hand side of (6) is a by-product of the proof of Theorem 1.7, which holds for arbitrary cusped manifolds. However, under some additional assumptions on M (for instance if M is “gentle” according to [2, 3]; then “ $a = \emptyset$ ”) we could avoid the delicate surgery argument, and define via more direct constructions (similar to those for partition functions) the invariants $\mathcal{H}_N(V, \rho_{\text{hyp}}, h, k)$ for arbitrary weights relative to the hyperbolic holonomy.

1.3. Enhanced QH state sums. The complications that appear when defining QHI of patterns based on cusped manifolds depended ultimately on a technical difficulty that we will overcome in the present paper, and that we are going to illustrate. For every topological pattern $Y_{\mathcal{P}}$, denote by \hat{V} the compact space obtained by filling each boundary component of ∂V with the cone over it. If $\partial V = \emptyset$ then $\hat{V} = W$. Note that \hat{V} has just a finite set of non-manifold points, that is the vertices of the filling cones. For every pattern \mathcal{P} , the quantum hyperbolic invariants $\mathcal{H}_N(\mathcal{P})$ are computed by state sums over certain “decorated” triangulations T of \hat{V} , depending on the choice of a point z_{ρ} in the associated gluing variety $G(T)$ such that z_{ρ} represents the character ρ (see Section 2), and on a suitable encoding of the weights (h, k) . Moreover, one requires that T is equipped with a *branching* b ; equivalently, T supports a structure of Δ -complex in the sense of [20].

In the case of QHFT partition functions this is not so demanding: T can be a “quasi-regular” triangulation (every edge has distinct endpoints), the branching b can be induced by a total ordering of the vertices, and z_{ρ} can be realized by means of a so-called “idealization” of $PSL(2, \mathbb{C})$ -valued 1-cocycles on (T, b) .

In the case of a cusped manifold M , we use *ideal* triangulations T of \hat{V} such that the gluing variety $G(T)$ contains a point z_h that represents the hyperbolic holonomy ρ_{hyp} and has coordinates (given by shape parameters of hyperbolic structures on the tetrahedra of T) with non-negative imaginary part (shortly we will say that z_h is *non-negative*). Such triangulations exist for every M , for instance we can obtain such pairs (T, z_h) by subdividing the canonical Epstein-Penner cell decomposition of M . However, we do not know if for every M there exists such a triangulation T that can be equipped with a branching b . For instance, the canonical Epstein-Penner decomposition of the figure-8-knot-sister cusped manifold is in fact a triangulation, made by two regular hyperbolic ideal tetrahedra, and

one can check that this triangulation does not admit any branching. The “gentle” manifolds M , for which we were able to define in [2] the invariants $\mathcal{H}_N(M, \rho_{\text{hyp}}, (h, k))$ for arbitrary weights, have by definition a branched, possibly somewhere negative, triangulation (T', b, z'_h) that nevertheless can be obtained in a *prescribed way* from a possibly non-branchable non-negative (T, z_h) .

In order to solve this complication and eventually get Theorem 1.1, we relax the requirement of branchings on triangulation by allowing a kind of *weak branching* that always exist, and by using *enhanced state sums* obtained by tracing suitable tensor networks carried by the triangulations. The key point will be to achieve the invariance of the values of the state sums under changes of the weak branching. To get it, besides the usual *matrix dilogarithm operators* associated to the tetrahedra, we will introduce also determined *2-face tensors*. This will work up to a slightly worse phase ambiguity (nevertheless, see Remark 6.12).

Remark 1.8. A more restrictive version of Theorem 1.1 holds true when M is *very gentle* by using the usual (non-enhanced) QH state sums, hence with the better phase ambiguity. All interesting consequences of the asymptotic results of [5] already appear with very gentle cusped manifolds.

1.4. Plan of the paper. In the order:

- In Section 2 we recall a few general facts about “structured” triangulations”, i.e. triangulations endowed with *pre-branchings*, *weak branchings*, or *branchings*, and the associated *gluing varieties*.
- Let V be as in Section 1.2, \hat{V} as in Section 1.3, (T, \tilde{b}) a weakly branched triangulation of \hat{V} , and c a *rough charge* on T (suitably specialized “global charges” will eventually encode the c -weights). In Section 3 we construct for every odd $N \geq 3$ the so-called *coarse analytic configuration* $\mathcal{A}_N^C(T, \tilde{b}, c)$. In particular it contains an infinite Abelian covering of the gluing variety $G(T)$,

$$p_\infty : G(T, \tilde{b})_\infty \rightarrow G(T)$$

and an analytic function

$$(7) \quad \mathcal{H}_N^e(T, \tilde{b}, c) : G(T, \tilde{b})_\infty \rightarrow \mathbb{C} .$$

In the case of QHFT patterns, the coarse analytic configurations are qualitatively described in the following informal statement.

Proposition 1.9. *For every topological support $Y = (V, L, (h_c, k_c))$ we can construct a weakly branched triangulation (T, \tilde{b}) of (\hat{V}, L) and a global charge c on T such that:*

- (1) *The global charge c determines the c -weight (h_c, k_c) .*
- (2) *For every QHFT pattern $\mathcal{P} = (V, L, \rho, (h, k))$ with topological support Y , there is a point $u \in G(T, \tilde{b})_\infty$ such that $p_\infty(u)$ represents the character ρ , and for every odd $N \geq 3$ we have*

$$\mathcal{H}_N^e(\mathcal{P}) \equiv_{4N} \mathcal{H}_N^e(T, \tilde{b}, c)(u) .$$

In the case of patterns based on a cusped manifold, we have:

Proposition 1.10. *For every topological support $Y = (V, (h_c, k_c))$ based on a cusped manifold M , there is a determined Zariski open subset $\Omega(M)$ of $X_0(M)$ containing ρ_{hyp} , and a weakly branched ideal triangulation (T, \tilde{b}) of \hat{V} and a global charge c on T such that:*

- (1) *The global charge c determines the c -weight (h_c, k_c) .*
- (2) *The gluing variety $G(T)$ contains a non-negative point z_h that represents the hyperbolic holonomy ρ_{hyp} .*
- (3) *There are an irreducible component Z of $G(T)$, a Zariski open subset Ω_Z of Z that contains z_h , and a regular rational isomorphism $\rho : \Omega_Z \rightarrow \Omega(M)$ such that for every $z \in \Omega_Z$, $\rho(z)$ is the character of V represented by z .*
- (4) *For every pattern $\mathcal{P} = (V, \rho, (h, k))$ based on M , with topological support Y and such that $\rho \in \Omega(M)$, there is a point $u \in Z_\infty := p_\infty^{-1}(Z)$ such that $\rho = \rho(p_\infty(u))$ and for every odd $N \geq 3$ we have*

$$\mathcal{H}_N^e(\mathcal{P}) \equiv_{4N} \mathcal{H}_N^e(T, \tilde{b}, c)(u) .$$

Remarks 1.11. (1) In the case of QHFT partition functions or when M is very gentle, we can use genuine branchings and get versions of the previous propositions using the non-enhanced QHI (defined up to multiplication by $2N$ th-roots of unity) and the corresponding analytic functions $\mathcal{H}_N(T, b, c)$ in the analytic configurations.

(2) We will have concrete models of the finite coverings and regular rational maps mentioned in (3) of Theorem 1.1 by considering a suitable factorization of $\mathcal{H}_N^e(T, \tilde{b}, c) : Z_\infty \rightarrow \mathbb{C}$.

- In Section 4 we develop the content of Proposition 1.10 and we prove Theorem 1.1.
- In Section 5 we indicate briefly how to deal similarly with the QHFT partition functions. In particular we extend to this case the *factorization* result of point (4) in Theorem 1.1.
- Section 6 contains all computations and technical results concerning the enhanced state sums.

Notations. We reserve the character N to odd integers ≥ 3 , and put $m := (N - 1)/2$. We set $\mathcal{I}_N = \{0, \dots, N - 1\}$, identified with $\mathbb{Z}/N\mathbb{Z}$ with its Abelian group structure, and denote by $[n]_N \in \mathcal{I}_N$ the residue modulo N of an integer n , valued in \mathcal{I}_N . We let $\delta_N(n) := 1$ (resp. 0) if $[n]_N = 0$ (resp. $[n]_N \neq 0$). The root of unity $\zeta_d = \exp(2\pi i d/N)$ is denoted by ζ if there is no ambiguity.

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2. STRUCTURED TRIANGULATIONS AND GLUING VARIETIES

Triangulations. Let V be as in Section 1.2, that is, a compact oriented connected 3-manifold with (possibly empty) boundary ∂V made by toric components. Denote by \hat{V} the space obtained by taking the cone over each boundary component of ∂V . A *triangulation* T of \hat{V} is a collection of *abstract oriented* tetrahedra $\Delta_1, \dots, \Delta_s$ together with a complete system \sim of pairings of their 2-faces via orientation reversing affine isomorphisms, such that the oriented quotient space

$$T := \coprod_{i=1}^s \Delta_i / \sim$$

is homeomorphic to \hat{V} , preserving the orientation. We will distinguish between the abstract 2-faces, edges and vertices of $\coprod_{i=1}^s \Delta_i$, and the ones of T after the 2-face pairings. In particular we denote by $E(\vec{\Delta})$ and $E(T)$ the set of edges of $\coprod_{i=1}^s \Delta_i$ and of T respectively. For all $E \in E(\vec{\Delta})$ and $e \in E(T)$ we write $E \rightarrow e$ to mean that E is sent to e under the 2-face pairings. Every (abstract) 2-face of each Δ_i inherits the *boundary orientation* by the rule: “*first the outgoing normal*”. Necessarily the non-manifold points of \hat{V} (when $\partial V \neq \emptyset$) are vertices of T . A triangulation T of \hat{V} is called *ideal* if the set of vertices of T coincides with the set of non-manifold points of \hat{V} .

Gluing varieties. Let T be a triangulation of \hat{V} as above. For every tetrahedron Δ_j choose a vertex v^j . Order cyclically the edges of the opposite 2-face F^j by the *opposite* of the boundary orientation of F^j . Fix an auxiliary compatible linear ordering E_0^j, E_1^j, E_2^j . Give each edge $E \in E(\vec{\Delta})$ an indeterminate $X(E)$ so that for each Δ_j opposite edges have the same indeterminate. In particular, put

$$X_r^j := X(E_r^j), \quad r \in \{0, 1, 2\}.$$

Set

$$\mathbb{C}_* := \mathbb{C} \setminus \{0, 1\}.$$

Define the algebraic set (the index r is considered mod(3))

$$G_3(T) = \{(X_0^1, X_1^1, X_2^1, \dots, X_0^s, X_1^s, X_2^s) \in \mathbb{C}_*^{3s} \mid \\ \forall j \in \{1, \dots, s\}, r \in \{0, 1, 2\}, X_{r+1}^j(1 - X_r^j) = 1, \\ \forall e \in E(T), \prod_{E \rightarrow e} X(E) = 1\}.$$

Denote by $G_2(T)$ the image of $G_3(T)$ under the projection $(\mathbb{C}_*^3)^s \rightarrow (\mathbb{C}_*^2)^s$ onto the first two factors, and similarly, by $G_1(T)$ the image of $G_3(T)$ under the projection $(\mathbb{C}_*^3)^s \rightarrow \mathbb{C}_*^s$ onto the first factor. By the first set of s “tetrahedral” equations, $G_3(T)$ is the graph of two explicit rational regular maps defined on $G_2(T)$ or on $G_1(T)$. So these algebraic varieties are canonically isomorphic to each other. Moreover, it is evident that also the auxiliary choices that we have made are immaterial up to canonical isomorphism of algebraic varieties. We denote by $G(T)$ the algebraic variety defined up to isomorphism by any of these concrete models.

It is well-known that every point u of $G(T)$ represents a $PSL(2, \mathbb{C})$ -character $\rho(u)$ of V , and that the entries u_r^j , $r = 0, 1, 2$, of u corresponding to Δ_j can be interpreted as the *cross-ratios* parameters of an isometry class of oriented hyperbolic ideal tetrahedra, (Δ_j, u^j) . The imaginary parts of the u_r^j have a same sign $\epsilon(u^j) \in \{0, \pm 1\}$. Hence (Δ_j, u^j) has a well defined *algebraic volume*, given by

$$(8) \quad \text{Vol}_{\text{alg}}(\Delta_j, u^j) := \epsilon(u^j) \text{Vol}(\Delta_j, u^j)$$

where on the right side Vol stands for the geometric positive volume (forgetting the orientation). By a classical result of Schläfli,

$$\text{Vol}_{\text{alg}}(\Delta_j, u^j) = D_2(u_0^j)$$

where D_2 is the *Bloch–Wigner dilogarithm*, a real analytic function on \mathbb{C}_* , continuous at 0 and 1 (see (23) for a formula). Note that when the u_r^j are real, (Δ_j, u^j) is degenerate and both sides of (8) vanish. By summing the algebraic volume over the tetrahedra of T one defines the *volume function* on the gluing variety

$$(9) \quad \text{Vol} : G(T) \rightarrow \mathbb{R}.$$

For every point $u \in G(T)$ we have

$$\text{Vol}(u) = \text{Vol}(\rho(u))$$

where on the right side Vol stands for the (intrinsically defined) volume of the character $\rho(u)$.

In general $G(T)$ might be empty. However, our topological supports $Y_{\mathcal{P}}$ always admit triangulations T of \hat{V} such that $G(T)$ is highly non trivial. A first general result concerns its dimension.

Theorem 2.1. ([32], [30]; see also [7]) *Assume that V has one torus boundary component. Let T be an ideal triangulation of \hat{V} . If the gluing variety $G(T)$ is non empty, then it is a complex algebraic curve.*

This result depends only on the combinatorial properties of the ideal triangulations of V . If moreover the interior of V is a cusped manifold M , then it has the canonical Epstein–Penner (EP) cell decomposition by embedded convex hyperbolic ideal polyhedra (see for instance [7]). Then we have:

Proposition 2.2. *The maximal subdivisions of the EP cell decomposition of M define a finite set $\mathcal{T}_{EP}(M)$ of ideal triangulations of \hat{V} , such that for every $T \in \mathcal{T}_{EP}(M)$ the gluing variety $G(T)$ contains a non-negative point $u_h \in G(T)$ such that $\rho(u_h) = \rho_{\text{hyp}}$, and $\text{Vol}(u_h) = \text{Vol}(M)$.*

Every non-degenerate hyperbolic ideal tetrahedron carried by such a (T, u_h) has non-negative volume, but in general one cannot avoid the presence of some degenerate tetrahedra. We call $\mathcal{T}_{EP}(M)$ the set of *EP-triangulations* of M .

For other manifolds the situation is not so well-known. Let us just consider the case when $V = W$ is compact closed. A triangulation T of W is called *quasi-regular* if every edge of T has distinct vertices in T . It is clear that every triangulation can be subdivided to become quasi-regular. Fix a total ordering on the vertices of T and orient every edge e of T from v to v' if $v < v'$. Every simplicial $PSL(2, \mathbb{C})$ -valued 1-cocycle z on T , defined by using such an edge orientation, represents a character

$\rho(z)$ of W . By an *idealization* procedure widely used in [1, 2, 3], we can associate to a sufficiently generic cocycle z a point $u \in G(T)$ such that $\rho(z) = \rho(u)$. Then we can prove:

Proposition 2.3. *Let T be a quasi-regular triangulation of W . For every character ρ of W there is a point $u \in G(T)$ such that $\rho = \rho(u)$.*

A similar, slightly more elaborated result holds in all the other cases of topological supports covered by the QHFT partition functions; it uses triangulations of \hat{V} obtained from quasi-regular relative triangulations $(T, \partial T)$ of $(V, \partial V)$ by adding a cone over each component of ∂T .

Clearly, the point u in Proposition 2.3 is far to be unique. For instance if $W = S^3$ every point of $G(T)$ represents the trivial character. The idealization procedure is reminiscent of Thurston's spinning construction, and is strictly related to it when W is hyperbolic. In that case, the following Proposition 2.4, which is proved by using the spinning construction, applies to quasi regular triangulations, hence it agrees with Proposition 2.3.

Proposition 2.4. ([28]) *Let T be a triangulation of a compact closed oriented hyperbolic 3-manifold W , with the property that every edge of T with only one vertex in T is essential, i.e. it is not null-homotopic in W . Then there exists $u \in G(T)$ such that $\text{Vol}(u) = \text{Vol}(W)$. Moreover, for any such u , $\rho(u) = \rho_{\text{hyp}}(W)$ is the hyperbolic holonomy.*

Variations on branched triangulations. Define a *pre-branched tetrahedron* (Δ, σ) as an oriented tetrahedron Δ endowed with a system σ of transverse 2-face co-orientations such that two transverse arrows are outgoing and two are ingoing. As every 2-face has the boundary orientation, by duality σ can be interpreted as a system of 2-face orientations.

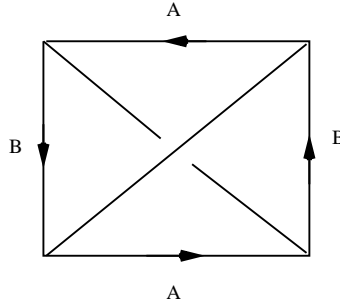


FIGURE 1. Pre-branched tetrahedron.

Figure 1 shows a pre-branched tetrahedron (Δ, σ) embedded in \mathbb{R}^3 , with coordinates (x_1, x_2, x_3) such that the plane of the picture is $\{x_3 = 0\}$. We stipulate that Δ inherits the ambient orientation, and that the two 2-faces intersecting at the over-crossing (resp. under-crossing) *diagonal edge* have outgoing (resp. ingoing) transverse arrows. Every *square edge* of Δ is naturally oriented as a common boundary edge of two 2-faces with opposite transverse co-orientations, so that the square edges form an oriented quadrilateral. We can also distinguish a pair of opposite A (resp. B) square edges, in a way which is reminiscent of the “ A vs. B crossing simplification” in the definition of the Kauffmann bracket. Note that we use the orientation of Δ in order to distinguish the A from the B edges. The orientation of the diagonal edges is undetermined. Up to isomorphism there is only one pre-branched tetrahedron.

An oriented tetrahedron Δ becomes a 3-simplex if we order its vertices, say v_0, v_1, v_2, v_3 . This can be encoded equivalently by the system b of orientations of the edges of Δ , called a (local) *branching*, so that v_j has j incoming edges. The 2-faces of (Δ, b) are ordered accordingly with the opposite vertices, and naturally inherit a branching b_F . The branchings b and b_F induce an orientation on Δ and its 2-faces, the *b -orientation*. It may coincide or not with the orientation of Δ . We encode this by a sign $*_b = \pm 1$. On two 2-faces the b_F -orientation coincides with the boundary orientation. Hence b naturally carries a pre-branching σ_b .

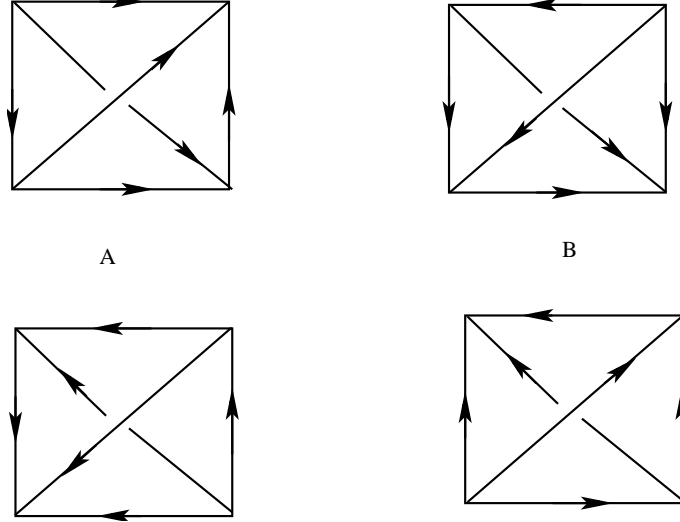


FIGURE 2. Branched tetrahedra inducing the same pre-branched tetrahedron.

On the other hand, given a pre-branching (Δ, σ) , there are four branched tetrahedra (Δ, b) such that $(\Delta, \sigma) = (\Delta, \sigma_b)$. They can be obtained by implementing the following procedure:

Choose an *A* (resp. *B*) square edge of (Δ, σ) and invert its orientation; then there is only one way to complete the so obtained square edge orientations to get (Δ, b) such that $(\Delta, \sigma) = (\Delta, \sigma_b)$. Note that:

- $*_b = 1$ (resp. $*_b = -1$) if and only if we have chosen an *A* (resp. *B*) square edge.
- With respect the ordering v_0, v_1, v_2, v_3 of the vertices of (Δ, b) , the chosen square edge is $[v_0, v_3]$.

In Figure 2 we illustrate the above procedure.

Let us consider now some kinds of (globally) structured triangulations of \hat{V} .

- A *pre-branched triangulation* (T, σ) of \hat{V} is a triangulation formed by pre-branched tetrahedra

$$\{(\Delta_j, \sigma_j)\}_{j=1, \dots, s}$$

such that the transverse co-orientations of the abstract 2-faces match under the 2-face pairings; we denote by σ the resulting *global pre-branching* on T .

- A *weakly-branched triangulation* (T, \tilde{b}) is a triangulation formed by branched tetrahedra

$$\{(\Delta_j, b_j)\}_{j=1, \dots, s}$$

such that the induced pre-branched tetrahedra $\{(\Delta_j, \sigma_{b_j})\}_{j=1, \dots, s}$ form a pre-branched triangulation (T, σ) . On the other hand, given any pre-branched triangulation (T, σ) we can convert every pre-branched tetrahedron into a compatible branched one to get a weakly-branched triangulation (T, \tilde{b}) that induces (T, σ) .

- A *branched triangulation* (T, b) is a triangulation formed by branched tetrahedra

$$\{(\Delta_j, b_j)\}_{j=1, \dots, s}$$

such that the local branchings match under the 2-face identifications; we denote by b the resulting *global branching* on T . Obviously a branched triangulation is weakly-branched.

Remark 2.5. The notion of branched triangulation of \hat{V} is equivalent to the notion of Δ -complex over \hat{V} in the sense of [20]. Note in particular that the 3-chain $\sum_j *_b(\Delta_j, b_j)$ represents the fundamental class in $H_3(\hat{V}; \mathbb{Z})$.

Given a triangulation T of \hat{V} , denote by \bar{V} the compact 3-manifold with boundary obtained from V by removing a small open 3-ball around every vertex of T that is a manifold point. Clearly, $V = \bar{V}$ if

and only if T is ideal. A pre-branched triangulation (T, σ) of \hat{V} can be described in a very concrete way in terms of the standard spine P of \bar{V} dual to T .

Lemma 2.6. *There is a natural bijection between the pre-branched triangulations (T, σ) supported by T and the so-called pre-branched standard spines $(P, \bar{\sigma})$ of \bar{V} , where $\bar{\sigma}$ is a system of edge orientations of the singular locus $\text{Sing}(P)$ such that at every vertex of P (dual to a tetrahedron of T) there are two outgoing and two ingoing germs of edges.*

The proof is evident, as every edge of $\text{Sing}(P)$ is transverse to one 2-face of T . The notion of (weakly) branched triangulation (T, \tilde{b}) has a natural dual counterpart as (weakly) branched spine (P, \bar{b}) .

Networks and \mathcal{N} -graphs. A weakly branched triangulation (T, \tilde{b}) of \hat{V} can be considered as a network of (abstract) branched tetrahedra

$$\mathcal{N} = \mathcal{N}(T, \tilde{b})$$

as follows. Place around every vertex v of the oriented graph $(\text{Sing}(P), \bar{\sigma})$ the dual abstract branched tetrahedron (Δ_v, b_v) in such a way that the edges of $\text{Sing}(P)$ at v and the 2-faces of (Δ_v, b_v) intersect transversely. Each edge of $\text{Sing}(P)$ connects now two abstract 2-faces which are identified in (T, \tilde{b}) : a 2-face F_i of the “initial” branched tetrahedron (Δ_{v_i}, b_{v_i}) with a 2-face F_f of the “final” one (Δ_{v_f}, b_{v_f}) . The identification of F_i with F_f can be encoded by a $\mathbb{Z}/3\mathbb{Z}$ edge coloring as follows. Let $S(J_3)$ be the symmetric group on the set $J_3 = \{0, 1, 2\}$, and $A(J_3)$ the subgroup of even permutations. Let us identify the group $A(J_3)$ with $(\mathbb{Z}/3\mathbb{Z}, +)$ via the isomorphism

$$\alpha : \mathbb{Z}/3\mathbb{Z} \rightarrow A(J_3), \quad \alpha(j) = (012)^j .$$

Let F be any abstract 2-face of some (Δ, b) , with the induced branching b_F . As usual, the vertices u_0^F, u_1^F, u_2^F of F are ordered accordingly to the branching b_F . Then, the gluing map ϕ_e of F_i onto F_f is determined by the permutation $\tau_e \in A(J_3)$ such that

$$\phi_e(u_j^i) = u_{\tau(j)}^f .$$

Give e the color $r(e) \in \mathbb{Z}/3\mathbb{Z}$ corresponding to τ_e under the isomorphism α . The network \mathcal{N} is defined as the oriented graph $(\text{Sing}(P), \bar{\sigma})$, where each vertex v is “structured” by the above correspondence between the germs of adjacent edges and the 2-faces of (Δ_v, b_v) , and each edge e has the $\mathbb{Z}/3\mathbb{Z}$ -color $r(e)$. The following two Lemmas are easy but important.

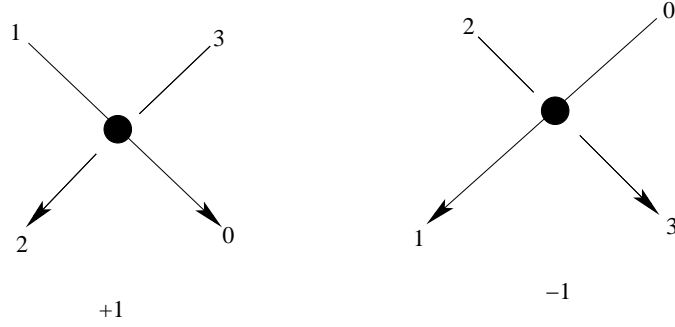
Lemma 2.7. *A weakly branched triangulation (T, \tilde{b}) is genuinely branched if and only if the $\mathbb{Z}/3\mathbb{Z}$ edge colors $r(e)$ of the associated network \mathcal{N} are constantly $r(e) = 0$.*

Lemma 2.8. *Every triangulation T of \hat{V} admits pre-branchings and hence compatible weak-branchings.*

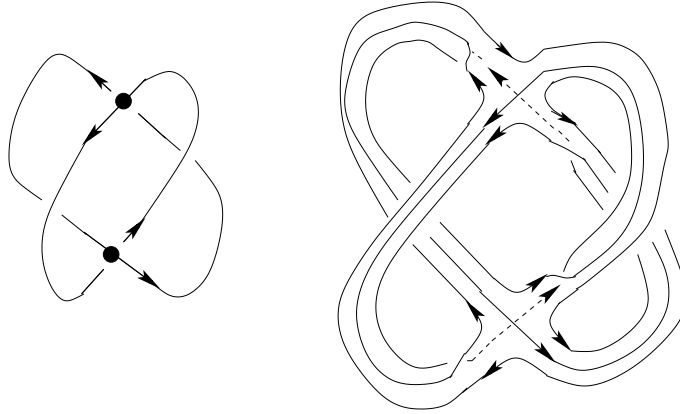
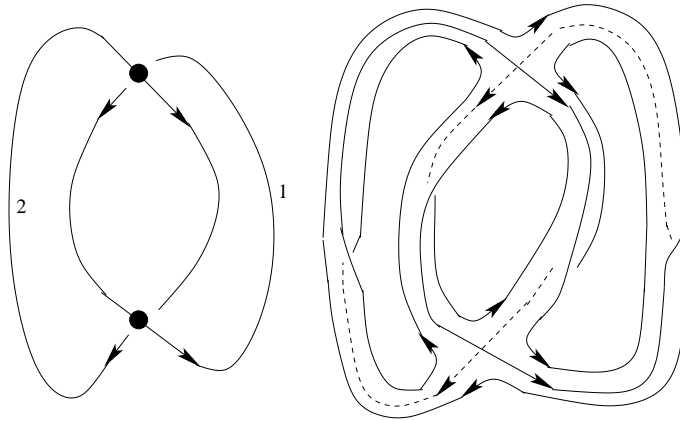
We stress that this is no longer true for genuine branchings (see the example in Figure 5). If T is quasi-regular, then every total order on its vertices induces a branching of T .

The following representation by means of suitable *planar diagrams*, called \mathcal{N} -graphs, is useful to deal with the networks $\mathcal{N}(T, \tilde{b})$ concretely. An \mathcal{N} -graph (Γ, θ) that represents $\mathcal{N}(T, \tilde{b})$ looks like an ordinary diagram of an *oriented link* (θ denotes the orientation). Some crossings are marked by a solid dot, while the others are “accidental” crossings. By definition, an (oriented) arc of Γ connects two (possibly coincident) dotted crossings. At first (Γ, θ) encodes an embedding in \mathbb{R}^3 of the oriented graph $(\text{Sing}(P), \bar{\sigma})$, so that the dotted crossings correspond to its vertices, and the arcs to its oriented edges. Moreover, every dotted crossing corresponding to a vertex v encodes the branched tetrahedron of (Δ_v, b_v) as we show in Figure 3: the ± 1 at the bottom refers to the signs $*_b$, while the label in $\{0, 1, 2, 3\}$ at each arc-germ refers to the name, accordingly with the b_v ordering, of the corresponding transverse 2-face of (Δ_v, b_v) . Finally every arc of (Γ, θ) corresponding to an edge e of $(\text{Sing}(P), \bar{\sigma})$ inherits the $\mathbb{Z}/3\mathbb{Z}$ -color $r(e)$. If $r(e) = 0$ we understand and hence omit it.

Given (Γ, θ) , an easy *decoding procedure* produces a concrete embedding in \mathbb{R}^3 of a closed regular neighbourhood $N(P)$ of $\text{Sing}(P)$ in the standard spine P . We believe that it is enough to show how all this works in the cases of the simplest cusped manifolds: M_0 , the figure-eight-knot complement in S^3 , and its “sister” M_1 , which is the complement of a knot in the lens space $L_{5,1}$ (see [17]). In

FIGURE 3. Decoding \mathcal{N} -graph crossings .

both cases the Epstein-Penner decomposition is a triangulation made by two regular hyperbolic ideal tetrahedra. We denote by T_0 and T_1 the underlying topological/combinatorial triangulations of \hat{V}_0 and \hat{V}_1 , respectively.

FIGURE 4. A \mathcal{N} -graph for (T_0, b) .FIGURE 5. A \mathcal{N} -graph for (T_1, \tilde{b}) .

In Figure 4 (resp. Figure 5) we show on the left side an \mathcal{N} -graph of a branched (T_0, b) (resp. weakly branched (T_1, \tilde{b})). It is not hard to verify that T_1 *does not carry any genuine branching*. On the right

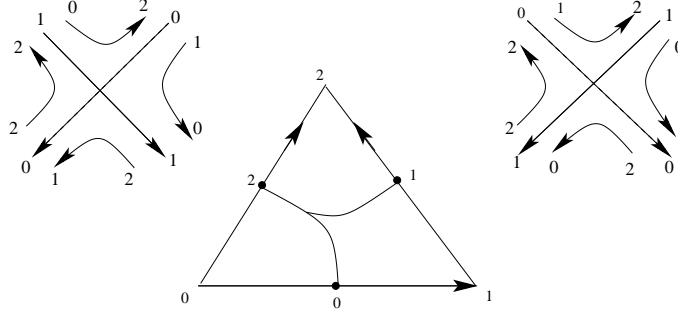
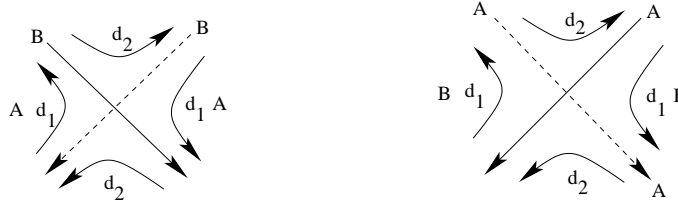


FIGURE 6.

side of the Figure we show the decoding of the \mathcal{N} -graph, that is, the associated embedding in \mathbb{R}^3 of (the boundary lines of) $N(P)$. Note that near each dotted crossing there are six arcs, that are locally oriented as portions of the boundary of the 2-regions of the spine P which are locally dual to the edges of the associated branched tetrahedron, and locally dually oriented by the weak branching. The endpoints of these boundary arcs are grouped in four sets of three points, one for each arc-germ e of the \mathcal{N} -graph. Each point of the triple associated to e belongs to the interior of one edge of the 2-face transverse to e , and is equipped with a label in J_3 as suggested in Figure 6.

FIGURE 7. Edge decorations on \mathcal{N} -graphs.

Edge decorations. We will use several instances of decorations of the abstract edges of a weakly branched triangulation, by stipulating that on every tetrahedron opposite edges have the same decoration. An example is given by decorating with A or B the square edges according to their type with respect to the associated pre-branching, and giving the empty color to the diagonal edges. Hence every such a decoration, d say, is determined by giving on each tetrahedron the triple of values $(d_0, d_1, d_2) := (d(E_0), d(E_1), d(E_2))$ (like the cross ratios (w_0, w_1, w_2) below). In terms of decoded \mathcal{N} -graphs, these values are placed as in Figure 7, where we understand that the over/under crossing boundary arcs are labelled by d_0 , and we show also the above A, B decoration.

Remark 2.9. In the case of genuine branchings, \mathcal{N} -graphs have been used in [4] under the name of *normal o-graphs* (here renamed *normal \mathcal{N} -graphs*). However, in [4] we used the opposite convention for the sign $*_b$, by adopting the usual crossing sign of link diagrams. By this choice we actually got the equality $H_N(S^3, L) \equiv_{2N} J_N(\bar{L})$, where \bar{L} is the mirror image of the link L . With the present (definitely preferable) convention we get the statement of Theorem 1.5.

The gluing varieties $G(T, \tilde{b})$. If (T, \tilde{b}) is a weakly branched triangulation, we can use the weak branching to fix the auxiliary choices in the models $G_r(T)$, $r \in \{1, 2, 3\}$, of the gluing variety $G(T)$ (with the exception of the ordering of the tetrahedra). On every branched tetrahedron (Δ_j, b_j) we fix $v^j = v_3^j$ and the opposite 2-face $F^j = F_3^j$. If the sign $*_j := *_b = 1$, then we order the edges of F^j by

$$E_0^j = [v_0^j, v_1^j], \quad E_1^j = [v_1^j, v_2^j], \quad E_2^j = [v_0^j, v_2^j].$$

If the sign $*_j = -1$, then we order the edges of F^j by

$$E_0^j = [v_0^j, v_2^j], \quad E_1^j = [v_1^j, v_2^j], \quad E_2^j = [v_0^j, v_1^j].$$

In both cases the linear order is compatible with the cyclic order induced by the opposite of the 2-face boundary orientation, according to our convention. Moreover, we use the weak branching to define the following rational regular automorphism of $(\mathbb{C}_*^3)^s$:

$$\begin{aligned} w : (\mathbb{C}_*^3)^s &\longrightarrow (\mathbb{C}_*^3)^s \\ u = (u_0^j, u_1^j, u_2^j)_{j=1, \dots, s} &\longmapsto w(u) := (w_0^j, w_1^j, w_2^j)_{j=1, \dots, s} \end{aligned}$$

such that

$$(10) \quad (w_0^j, w_1^j, w_2^j) = \begin{cases} (u_0^j, u_1^j, u_2^j) & \text{if } *_j = 1 \\ (1/u_2^j, 1/u_1^j, 1/u_0^j) & \text{if } *_j = -1 \end{cases}.$$

We will eventually use the images by w of the varieties $G_r(T)$, $r \in \{1, 2, 3\}$, obtained as above by using \tilde{b} . We denote them $G_r(T, \tilde{b})$. For every branched tetrahedron (Δ_j, b_j) , in both cases $*_j = \pm 1$ we order now the edges of F_3^j by

$$E_0^j = [v_0^j, v_1^j], \quad E_1^j = [v_1^j, v_2^j], \quad E_2^j = [v_0^j, v_2^j]$$

and we set

$$w_i^j = w(E_i^j), \quad w^j = (w_0^j, w_1^j, w_2^j).$$

Then the point $w = w(u)$ of $G_r(T, \tilde{b})$ determines a new system of cross ratios on the branched tetrahedra (Δ_j, b_j) . The defining equation of $G(T, \tilde{b})$ associated to an edge e of T is

$$\prod_{E \rightarrow e} w(E)^{*E} = 1$$

where if E is an edge of (Δ_j, b_j) we have $w(E) = w_i^j$ if and only if E is E_i^j or the opposite edge, and $*E = *_j$. The volume function on $G(T, \tilde{b})$ takes now the form

$$\text{Vol}(w) = \sum_j *_j D_2(w_0^j).$$

Clearly $\text{Vol}(u) = \text{Vol}(w(u))$.

Example 2.10. It is easy to recover the edge equations of $G(T, \tilde{b})$ from a representing decoded \mathcal{N} -graph. One places the cross-ratio variables (w_0, w_1, w_2) at each dotted crossing as in Figure 7, and take the product of these variables occurring along each boundary line of $N(P)$. For example, consider the cusped manifold M_1 and (T_1, \tilde{b}) . Give the indeterminates (w_0, w_1, w_2) to the top crossing of Figure 5, and (W_0, W_1, W_2) to the bottom one. Note that both have $*_b = 1$. By taking the product of these indeterminates along the boundary components of the two regions of P , we get the two equations

$$(11) \quad w_0 w_1^2 W_0 W_1^2 = 1, \quad w_0 w_2^2 W_0 W_2^2 = 1.$$

By using the relation $w_{j+1} = 1/(1 - w_j)$ and the similar one for W_j (j is considered mod(3)), they reduce to the unique quadratic equation

$$w_1(w_1 - 1)W_1(W_1 - 1) = 1.$$

We can solve it and get $W := W_1$ as a function of $w := w_1$. According to [17], the parameter space of “positive solutions” is the half plane $\text{Im } w > 0$ with the ray $0.5 + si$, $\sqrt{15}/2 \leq s < +\infty$, removed. The complete hyperbolic structure corresponds to $w = W = \exp(i\pi/3)$, hence it is realized by two regular ideal tetrahedra. Similarly, by using (T_0, b) we can recover Thurston’s celebrated treatment of the figure-eight-knot complement M_0 .

3. COARSE ANALYTIC CONFIGURATIONS

In this section we build the so-called *coarse analytic configurations* associated to every weakly-branched triangulation (T, \tilde{b}) of \hat{V} . First we will describe the local configurations over a single branched tetrahedron.

3.1. Local analytic configurations. Take an oriented 3-simplex (Δ, b) . As in the previous section, select the 2-face F_3 opposite to the vertex v_3 . The edges of F_3 are ordered as

$$E_0 = [v_0, v_1], \quad E_1 = [v_1, v_2], \quad E_2 = [v_0, v_2].$$

The local analytic configurations over (Δ, b) are families of spaces and maps, indexed by the odd integers $N \geq 3$, associated to certain decorations of (Δ, b) that turn it into a *quantum hyperbolic 3-simplex*, together with *tetrahedral tensors*. We describe them in the next two subsections, and then give the formal definition of the local analytic configurations over (Δ, b) .

3.1.1. Quantum hyperbolic 3-simplices. First we are going to define further edge decorations. We stipulate once and for all that each of them gives the same decoration to opposite edges. Hence they will be specified by triples, say $d = (d_0, d_1, d_2)$, where $d_r = d(E_r)$, $r \in \{0, 1, 2\}$.

A *quantum hyperbolic 3-simplex* is a tuple (Δ, b, w, f, c) where:

- $w = (w_0, w_1, w_2)$ is the system of cross ratios (10).
- $f = (f_0, f_1, f_2) \in \mathbb{Z}^3$ satisfies $l_0 + l_1 + l_2 = 0$, where for every $k = 0, 1, 2$ we set (\log denotes the branch of logarithm with imaginary part in $(-\pi, \pi]$):

$$(12) \quad l_k = \log(w_k) + if_k \pi.$$

- $c = (c_0, c_1, c_2) \in \mathbb{Z}^3$ satisfies $c_0 + c_1 + c_2 = 1$.

Finally, for every $k = 0, 1, 2$ we set

$$(13) \quad l_{k,N,*_b,c} = \frac{1}{N}(\log(w_k) + i\pi(N+1)(f_k - *_b c_k)).$$

We call f a (local) *flattening* on (Δ, b, w) , c a (local) *charge* on (Δ, b) , and l_k and $l_{k,N,*_b,c}$ a (local) *classical log-branch* and a (level N) *quantum log-branch* respectively. Note that c does not depend on w . However, if w has positive (resp. negative) imaginary parts, then c is a local charge if and only if $-c$ (resp. c) is a local flattening.

Now, we are going to organize these decorations as they appear in the local analytic configurations. Denote by

$$p_\infty : \mathcal{W}_\infty \rightarrow \mathbb{C}_*$$

the maximal abelian covering of \mathbb{C}_* . It is explicitly given by

$$(14) \quad \mathcal{W}_\infty = (\mathcal{D} \times \mathbb{Z}^2) / \sim$$

where \mathcal{D} is obtained by adding four open half-lines to the boundary of $\mathbb{C} \setminus ((-\infty, 0) \cup (1, +\infty))$, and

$$(15) \quad \begin{aligned} (x + i0; p, q) &\sim (x - i0; p + 2, q) & \text{if } x \in (-\infty, 0) \\ (x + i0; p, q) &\sim (x - i0; p, q + 2) & \text{if } x \in (1, +\infty). \end{aligned}$$

We have the natural projection $p_\infty([z; p, q]) = z$. One can identify \mathcal{W}_∞ with the Riemann surface of the maps

$$(16) \quad \phi_{\epsilon, \epsilon'} : x \rightarrow (\log(x) + i\epsilon\pi, \log((1-x)^{-1}) + i\epsilon'\pi)$$

where $\epsilon, \epsilon' \in \{0, 1\}$. Indeed, set $l(y; m) := \log(y) + im\pi$, $m \in \mathbb{Z}$. Then the bijective map

$$(17) \quad (l(x; p), l((1-x)^{-1}; q)) \rightarrow (x; \frac{l(x; p) - \log(x)}{i\pi}, \frac{l((1-x)^{-1}; q) - \log((1-x)^{-1})}{i\pi})$$

realizes such an identification.

Fix now $c = (c_0, c_1) \in \mathbb{Z}^2$. For every odd $N \geq 3$ and every sign $* = \pm 1$ define the analytic map

$$l_{N,*_b,c} : \mathcal{W}_\infty \rightarrow (\mathbb{C}^*)^2$$

$$l_{N,*_b,c}([x; p, q]) = (\frac{1}{N}(\log(x) + i\pi(N+1)(p - *_b c_0)), \frac{1}{N}(\log((1-x)^{-1}) + i\pi(N+1)(q - *_b c_1))) .$$

Recall that we have fixed a 3-simplex (Δ, b) . We have:

- As any triple of cross ratios (w_0, w_1, w_2) is determined by w_0 , \mathbb{C}_* is a natural space of parameters for the decorations (Δ, b, w) of (Δ, b) . Similarly the points $[w_0; f_0, f_1] \in \mathcal{W}_\infty$ are parameters for the decorations (Δ, b, w, f) .

- Every fixed couple $(c_0, c_1) \in \mathbb{Z}^2$ determines a charge $c = (c_0, c_1, c_2)$ on (Δ, b) . Then the components of $l_{N,*,b,c}([w_0; f_0, f_1])$ give the (level N) quantum log-branches (13) of (Δ, b, w, f, c) for $k = 0, 1$, and hence also $l_{2,N,*,b,c} = -l_{0,N,*,b,c} - l_{1,N,*,b,c}$.
- Consider the map

$$\text{EXP} \circ l_{N,*,c} : \mathcal{W}_\infty \rightarrow (\mathbb{C}^*)^2$$

where for every $s \geq 1$ we put $\text{EXP} : (\mathbb{C}^*)^{2s} \rightarrow (\mathbb{C}^*)^{2s}$, $\text{EXP}(z_1, \dots, z_{2s}) = (\exp(z_1), \dots, \exp(z_{2s}))$. For every $k = 0, 1, 2$ set

$$(18) \quad w'_k = \exp(l_{k,N,*,b,c}([w_0; f_0, f_1])) .$$

Clearly $(w'_k)^N = w_k$. Hence we get a distinguished system of N th-roots of the cross ratios w_k .

The main properties of the above edge decorations are summarized in the following Lemma; the proof is straightforward, anyway see [2] and [3], Remarks 2.1 and 2.18.

Lemma 3.1. (1) The N th-roots (w'_0, w'_1, w'_2) depend only on the residues mod (N) of the flattening f and the charge c , and verify the relation

$$(19) \quad w'_0 w'_1 w'_2 = -\zeta_N^{-*b(m+1)} = -\zeta_N^{*bm}$$

where as usual $N := 2m + 1$ and $\zeta_N := \exp(2i\pi/N)$.

(2) For any charge c and any system (u_0, u_1, u_2) of N th-roots of the cross ratios w_k verifying the relation (19), there is a flattening f such that $(u_0, u_1, u_2) = (w'_0, w'_1, w'_2)$ as in (18).

(3) The image \mathcal{W}_N of \mathcal{W}_∞ via the map $\text{EXP} \circ l_{N,*,b,c}$ consists of the curve of points $(u_0, u_1) \in (\mathbb{C}^*)^2$ such that

$$u_0^N + (u_1^{-1})^N = 1.$$

There is a natural N^2 -to-1 rational regular map $p_N : \mathcal{W}_N \rightarrow \mathbb{C}_*$, $p_N(u_0, u_1) := u_0^N$.

3.1.2. Tetrahedral tensors. At first we explain how for every integer $N \geq 1$, every operator $A \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$, expressed by means of its matrix elements in the standard basis of $\mathbb{C}^N \otimes \mathbb{C}^N$, is “carried” by a branched tetrahedron (Δ, b) . We stipulate that if $*_b = 1$ the matrix elements are indicated by $A_{k,l}^{i,j}$, $i, j, k, l \in \{0, \dots, N-1\}$, and by $A_{i,j}^{k,l}$ if $*_b = -1$. Associate to the 2-face F_j , $j \in \{0, 1, 2, 3\}$, a copy V_j of \mathbb{C}^N . Then put

$$(20) \quad A = \begin{cases} (A_{k,l}^{i,j}) : V_3 \otimes V_1 \rightarrow V_2 \otimes V_0 & \text{if } *_b = 1 \\ (A_{i,j}^{k,l}) : V_2 \otimes V_0 \rightarrow V_3 \otimes V_1 & \text{if } *_b = -1 \end{cases} .$$

Hence, in both cases $*_b = \pm 1$, the pair of indices (i, j) (resp. (k, l)) corresponds to the ordered couple (V_2, V_0) (resp. (V_3, V_1)), and (V_2, V_0) corresponds to the outgoing (resp. ingoing) arrows of the pre-branching induced by b if $*_b = 1$ (resp. $*_b = -1$).

Now, we describe the tetrahedral tensors involved in the local analytic configurations over (Δ, b) . By definition, an *enhanced quantum hyperbolic 3-simplex* (Δ, b, w, f, r) is endowed with a further 2-face decoration $r \in (\mathbb{Z}/3\mathbb{Z})^2$, where

$$r = (r_2, r_0), \text{ if } *_b = 1$$

$$r = (r_3, r_1), \text{ if } *_b = -1$$

and in both cases r labels the 2-faces with outgoing arrows. Then, the tetrahedral tensors are defined by

$$(21) \quad \begin{aligned} R_0(\Delta, b, w, f) &:= R_{(0,*,b)}([w_0; f_0, f_1]) \\ \mathcal{R}_N(\Delta, b, w, f, c) &:= \mathcal{R}_{(N,*,b,c)} \circ \text{EXP} \circ l_{N,*,b,c}([w_0; f_0, f_1]) \\ \mathcal{R}_N^e(\Delta, b, w, f, c, r) &:= \mathcal{R}_{(N,*,b,c,r)}^e \circ \text{EXP} \circ l_{N,*,b,c}([w_0; f_0, f_1]) \end{aligned}$$

in terms of maps

$$R_{(0,*)} : \mathcal{W}_\infty \rightarrow \text{GL}(\mathbb{C} \otimes \mathbb{C}) \cong \mathbb{C}^*$$

$$\mathcal{R}_{(N,*,c)} : \mathcal{W}_N \rightarrow \text{GL}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

$$\mathcal{R}_{(N,*,c,r)}^e : \mathcal{W}_N \rightarrow \text{GL}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

where $*$ = ± 1 , $c \in \mathbb{Z}^2$, and $N \geq 3$ is odd. Note that R_0 does not depend on the charge c and the color r . We will see that \mathcal{R}_N is a specialization of \mathcal{R}_N^e : in case $r = (0, 0)$ we have

$$\mathcal{R}_{(N,*,c,0)}^e = \mathcal{R}_{(N,*,c)}.$$

The map $R_{(0,*)}$ is analytic. It is the exponential of a suitable normalization of a uniformization $\text{mod}(\pi^2\mathbb{Z})$ of the *Rogers dilogarithm* (the normalization will kill the ambiguity). The tensors $\mathcal{R}_{(N,*,c)}$ (resp. $\mathcal{R}_{(N,*,c,r)}^e$) are regular rational maps, called *level N (resp. enhanced) matrix dilogarithms*. They verify a non commutative version of the fundamental dilogarithm five term functional identity (see [2]).

Formulas follow. We have

$$(22) \quad R_{(0,*)}([x; p, q]) := \exp \left(* \frac{2}{i\pi} \left(L(x) + \frac{i\pi}{2} (p \log(1-x) + q \log(x)) \right) \right)$$

where L is the *Rogers dilogarithm*, defined and analytic over $\mathcal{D} = \mathbb{C} \setminus \{(-\infty; 0) \cup (1; +\infty)\}$ by

$$L(x) = -\frac{\pi^2}{6} - \frac{1}{2} \int_0^x \frac{\log(t)}{1-t} + \frac{\log(1-t)}{t} dt.$$

The constant $-\pi^2/6$ yields $L(1) = 0$. L is related to the *Bloch-Wigner dilogarithm* D_2 , and hence to the volume of hyperbolic ideal tetrahedra (see Section 2), by

$$(23) \quad L(x) = -\frac{\pi^2}{6} + \frac{1}{2} \log(x) \log(1-x) + \text{Li}_2, \quad D_2(x) = \text{Im}(\text{Li}_2(x)) + \arg(1-x) \log|x|$$

where $\text{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t} dt$ is the *Euler dilogarithm*.

Next we turn to the map $\mathcal{R}_{(N,*,c)}$. For every $x \in \mathbb{C}^*$, put $x^{1/N} := \exp(\log(x)/N)$, extended to $0^{1/N} := 0$ by continuity. Set

$$[x] := N^{-1} \frac{1-x^N}{1-x}, \quad g(x) := \prod_{j=1}^{N-1} (1-x\zeta^{-j})^{j/N}, \quad h(x) := g(x)/g(1).$$

Consider the rational function ω depending on $n \in \mathbb{N}$ and defined on the Fermat plane curve

$$\mathcal{F}_N = \{(u')^N + (v')^N = 1\}$$

by $\omega(u', v'|0) := 1$ and

$$\omega(u', v'|n) := \prod_{j=1}^{[n]_N} \frac{v'}{1-u'\zeta^j}.$$

The function ω has period N in its integer argument, and g is analytic on the cut complex plane $\mathbb{C} \setminus \{r\zeta^k, r \geq 1, k = 1, \dots, N-1\}$. Define the regular rational map

$$\mathcal{L}_N : \mathcal{F}_N \rightarrow \text{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

by the entries (recall that $m = (N-1)/2$):

$$(24) \quad \mathcal{L}_N(u', v')_{k,l}^{i,j} = h(u') \zeta^{kj+(m+1)k^2} \omega(u', v'|i-k) \delta_N(i+j-l).$$

We can compute the entries of the inverse tensor at every point $(u', v') \in \mathcal{F}_N$:

$$(\mathcal{L}_N(u', v')^{-1})_{i,j}^{k,l} = \frac{[u']}{h(u')} \zeta^{-kj-(m+1)k^2} \frac{\delta_N(i+j-l)}{\omega(u'/\zeta, v'|i-k)}.$$

Then

$$(25) \quad \begin{aligned} \mathcal{R}_{(N,1,c)}(u_0, u_1)_{k,l}^{i,j} &:= (u_0^{-c_1} u_1^{c_0})^{\frac{N-1}{2}} \mathcal{L}_N(u_0, u_1^{-1})_{k,l}^{i,j} \\ \mathcal{R}_{(N,-1,c)}(u_0, u_1)_{i,j}^{k,l} &:= (u_0^{-c_1} u_1^{c_0})^{\frac{N-1}{2}} (\mathcal{L}_N(u_0, u_1^{-1})^{-1})_{i,j}^{k,l}. \end{aligned}$$

Note that the dependence of $\mathcal{R}_{(N,*,c)}$ on the charge c is entirely concentrated in the scalar factor $(u_0^{-c_1} u_1^{c_0})^{\frac{N-1}{2}}$, which is the same for both values of the sign $*$ = ± 1 . On the other hand, the

associated matrix dilogarithm $\mathcal{R}_N(\Delta, b, w, f, c)$ depends on c via both this scalar factor and the maps $l_{N,*b,c}$.

Finally, we come to the map $\mathcal{R}_{(N,*c,r)}^e$. For every odd $N \geq 3$, let S and T be the *symmetric* $N \times N$ matrices given by

$$S_i^j = N^{-1/2} \zeta^{ij} \quad , \quad T_i^j = \zeta^{i^2(m+1)} \delta_N(i+j).$$

The next Lemma (a proof is in Section 6) shows that the matrices S and T define an N -dimensional projective representation of $SL(2, \mathbb{Z})$, and that the matrix

$$\mathcal{Q} := S^{-1}T$$

is projectively of order 3, with projective ambiguity given by 4th roots of unity.

Lemma 3.2. *The matrices S and T satisfy the relations:*

- (1) $S^4 = I_N$
- (2) $(ST)^3 = \phi_N S^2$ where $\left(\frac{a}{b}\right)$ denotes the Legendre-Jacobi quadratic symbol, and $\phi_N \in \{\pm 1, \pm i\}$ is given by

$$\phi_N = \begin{cases} \left(\frac{m+1}{N}\right) & \text{if } N \equiv 1 \pmod{4} \\ \left(\frac{m+1}{N}\right) i & \text{if } N \equiv 3 \pmod{4} \end{cases}$$

- (3) $(S^{-1}T)^3 = (TS^{-1})^3 = \phi_N I_N$.

Then we set:

$$(26) \quad \begin{aligned} \mathcal{R}_{(N,1,c,r)}^e(u_0, u_1)_{k,l}^{i,j} &:= \sum_{u,v=0,\dots,N-1} \mathcal{R}_{N,1,c}(u_0, u_1)_{k,l}^{u,v} (\mathcal{Q}^{-r_2})_u^i (\mathcal{Q}^{-r_0})_v^j \\ \mathcal{R}_{(N,-1,c,r)}^e(u_0, u_1)_{i,j}^{k,l} &:= \sum_{u,v=0,\dots,N-1} \mathcal{R}_{N,-1,c}(u_0, u_1)_{i,j}^{u,v} (\mathcal{Q}^{-r_3})_u^k (\mathcal{Q}^{-r_1})_v^l. \end{aligned}$$

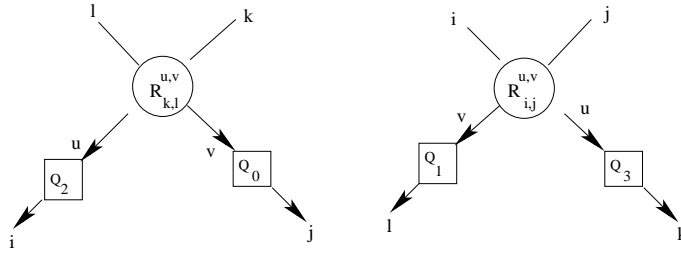


FIGURE 8. Enhanced matrix dilogarithms.

As already remarked above, we have in particular $\mathcal{R}_{(N,*c,r)}^e = \mathcal{R}_{(N,*c)}$ when $r = (0, 0)$. In Figure 8, referring to Figure 3, we show a graphical representation of these tensor identities, where $R_{c,d}^{a,b}$ stands for the non-enhanced matrix dilogarithm $\mathcal{R}_{(N,*c)}$ and Q_s for the power $\mathcal{Q}^{-r(s)}$.

3.1.3. The local analytic configurations over (Δ, b) . Summing up the two previous sections, we define the local analytic configurations over (Δ, b) as the families of spaces and maps, indexed by the odd integers $N \geq 3$, given by

$$\mathcal{A}_N(\Delta, b, c, r) := \{\mathcal{W}_\infty, \mathcal{W}_N, p_\infty, p_N, l_{N,*b,c}, R_{(0,*b)}, \mathcal{R}_{(N,*b,c,r)}^e\}.$$

for varying charges c and colors r on (Δ, b) .

3.2. Rough globalization. Let (T, \tilde{b}) be a weakly branched triangulation of \hat{V} formed by s tetrahedra. Consider the product space \mathcal{W}_∞^s . Fix a *rough global charge* $c \in (\mathbb{Z}^2)^s$, that is, just a collection of local charges c^j on the branched tetrahedra (Δ_j, b_j) . Then, for every point $[w; f] \in \mathcal{W}_\infty^s$ we have a so-called *rough QH enhanced triangulation* $(T, \tilde{b}, w, f, c, r)$, made by the enhanced quantum hyperbolic 3-simplices $(\Delta_j, b_j, w^j, f^j, c^j, r^j)$ where r^j labels a 2-face with outgoing transverse edge e of $(\text{Sing}(P), \hat{\sigma})$ with the $\mathbb{Z}/3\mathbb{Z}$ color $r(e)$. If $\tilde{b} = b$ is a branching, we have a *rough QH triangulation* (T, b, w, f, c) . We have also product maps (keeping, with slight abuse, the same notations)

$$\begin{aligned} p_\infty : \mathcal{W}_\infty^s &\rightarrow \mathbb{C}_*^s \\ p_N : \mathcal{W}_N^s &\rightarrow \mathbb{C}_*^s \\ l_{N,*,c} : \mathcal{W}_\infty^s &\rightarrow (\mathbb{C}^*)^{2s} \end{aligned}$$

and

$$\text{EXP} \circ l_{N,*,c} : \mathcal{W}_\infty^s \rightarrow \mathcal{W}_N^s \subset (\mathbb{C}^*)^{2s}$$

which coincide component by component with the corresponding local maps defined in Section 3.1, taking the signs $*_j = *_{b_j}$.

3.2.1. State sums. The globalization of the maps

$$\begin{aligned} R_{(0,*)} : \mathcal{W}_\infty &\rightarrow \mathbb{C}^* \\ \mathcal{R}_{(N,*,c)} \circ \text{EXP} \circ l_{N,*,c} : \mathcal{W}_\infty &\rightarrow \text{GL}(\mathbb{C}^N \otimes \mathbb{C}^N) \\ \mathcal{R}_{(N,*,c,r)}^e \circ \text{EXP} \circ l_{N,*,c} : \mathcal{W}_\infty &\rightarrow \text{GL}(\mathbb{C}^N \otimes \mathbb{C}^N) \end{aligned}$$

gives rise to *state sum* analytic functions

$$\begin{aligned} \mathcal{H}_0(T, \tilde{b}) : \mathcal{W}_\infty^s &\rightarrow \mathbb{C} \\ \mathcal{H}_N(T, b, c) : \mathcal{W}_\infty^s &\rightarrow \mathbb{C} \\ \mathcal{H}_N^e(T, \tilde{b}, c) : \mathcal{W}_\infty^s &\rightarrow \mathbb{C} \end{aligned}$$

defined as follows. Recall from Section 2 that (T, \tilde{b}) can be interpreted as an oriented graph \mathcal{N} with “structured” vertices and $\mathbb{Z}/3\mathbb{Z}$ -colored edges. For every point $[w; f] \in \mathcal{W}_\infty^s$ and every rough global charge c , associate to the vertex of \mathcal{N} dual to the branched tetrahedron (Δ_j, b_j) the operator

$$A(\Delta_j, b_j, w^j, f^j, c^j, r^j) \in \text{GL}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

given by

$$A(\Delta_j, b_j, w^j, f^j, c^j, r^j) = \begin{cases} R_0(\Delta_j, b_j, w^j, f^j) & \text{if } N = 1 \\ \mathcal{R}_N^e(\Delta_j, b_j, w^j, f^j, c^j, r^j) & \text{if } N = 2m + 1 \geq 3. \end{cases}$$

If $\tilde{b} = b$ is a branching, r is constantly 0 and so the tensor specializes to

$$A(\Delta_j, b_j, w^j, f^j, c^j, r^j) = \mathcal{R}_N(\Delta_j, b_j, w^j, f^j, c^j) \quad \text{if } N = 2m + 1 \geq 3.$$

This collection of tensors, one for each vertex of the graph \mathcal{N} , defines a *tensor network*. A N -state is an assignment of a label in $\{0, \dots, N-1\}$ to each edge of \mathcal{N} . Every N -state σ determines a matrix element $A(\Delta_j, b_j, w^j, f^j, c^j, r^j)_\sigma$, $j = 1, \dots, s$. Then, by tracing over all states and tetrahedra we get scalar state sums

$$(27) \quad S(T, \tilde{b}, c)([w; f]) := \sum_\sigma \prod_{j=1}^s A(\Delta_j, b_j, w^j, f^j, c^j, r^j)_\sigma.$$

Finally we put

$$(28) \quad S(T, \tilde{b}, c)([w; f]) = \begin{cases} \mathcal{H}_0(T, \tilde{b})([w; f]) & \text{if } N = 1 \\ \mathcal{H}_N^e(T, \tilde{b}, c)([w; f]) & \text{if } N = 2m + 1 \geq 3. \end{cases}$$

If $\tilde{b} = b$ is a branching, the state sum specializes to

$$(29) \quad S(T, b, c)([w; f]) = \mathcal{H}_N(T, b, c)([w; f]) \quad \text{if } N = 2m + 1 \geq 3.$$

Though $\mathcal{H}_0(T, \tilde{b})([w; f])$ is just the product of the scalars $R_0(\Delta_j, b_j, w^j, f^j)$, $j = 1, \dots, s$, we stress that it has the same formal definition as $\mathcal{H}_N^e(T, \tilde{b}, c)([w; f])$. Note also that

$$(30) \quad \mathcal{H}_N^e(T, \tilde{b}, c) = (\mathcal{H}^e)'_N \circ (\text{EXP} \circ l_{N, *, b, c})$$

where $(\mathcal{H}^e)'_N$ is a rational regular function defined on \mathcal{W}_N^s . Sometimes we will use instead the notations

$$\mathcal{H}_N^e(T, \tilde{b}, w, f, c) := \mathcal{H}_N^e(T, \tilde{b}, c)([w; f]) .$$

Also, when the context is clear, we write \mathcal{H}_0 for $\mathcal{H}_0(T, \tilde{b})$ and \mathcal{H}_N^e for $\mathcal{H}_N^e(T, \tilde{b}, c)$.

Remark 3.3. The enhanced state sums can be equivalently considered as combinations of *non-enhanced* matrix dilogarithms at the vertices, and tensors of the form $\mathcal{Q}^{-r(e)}$ at edges. In the case of branchings, the edge tensors are the identity, whence immaterial.

Remark 3.4. Below we refine the rough globalization by specifying subsets of \mathbb{C}_*^s , \mathcal{W}_∞^s , and \mathcal{W}_N^s , and taking the restriction of the maps and functions defined so far. However, we stress that many points of the asymptotic analysis hold already at the rough level (see [5]). Also, a rough tuple (T, \tilde{b}, w, f, c) is not completely meaningless. For instance, if all cross ratios $u = w^{*b}$ have strictly positive imaginary parts, then they define a (positive) hyperbolic structure on the complement of the 1-skeleton of T . The $*_b$ -signed sum of the Bloch-Wigner dilogarithms of the w_0^j 's is a well defined analytic function that computes the volume of such structures. Similarly, the above functions $\mathcal{H}_0(T, \tilde{b})$ and $\mathcal{H}_N^e(T, \tilde{b}, c)$ are perfectly well defined and analytic on rough structures.

3.3. Coarse globalization. Let (T, \tilde{b}) be as above. Consider the gluing variety $G(T, \tilde{b})$ defined at the end of Section 2, more precisely its concrete model $G_1(T, \tilde{b}) \subset \mathbb{C}_*^s$. Set

$$G(T, \tilde{b})_\infty := p_\infty^{-1}(G(T, \tilde{b})), \quad G(T, \tilde{b}, c)_N := \text{EXP} \circ l_{N, *, b, c}(G(T, \tilde{b})_\infty) .$$

By means of the respective restrictions of p_∞ and p_N (we will always keep the same notations) we get an infinite covering and a $(N^2)^s$ -covering of $G(T, \tilde{b})$. Then $G(T, \tilde{b})_\infty$ is a closed analytic subset of \mathcal{W}_∞^s , while $G(T, \tilde{b}, c)_N$ is an algebraic subset of \mathcal{W}_N^s . Note that

$$\text{Sing}(G(T, \tilde{b})_*) = p_*^{-1}(\text{Sing}(G(T, \tilde{b}))) .$$

Clearly the (restricted) maps p_∞ , p_N and $\text{EXP} \circ l_{N, *, b, c}$ form a commutative diagram.

For every point $[w; f] \in G(T, \tilde{b})_\infty$ and every rough global charge c , consider the corresponding rough tuple (T, \tilde{b}, w, f, c) . By using the notations introduced at the end of Section 2, for every edge e of T define its *total edge cross-ratio* by

$$W(e) := \prod_{E \rightarrow e} w(E)^{*E}$$

so that $[w; f] \in G(T, \tilde{b})_\infty$ if and only if for every edge e , $W(e) = 1$. Similarly, define the *total edge classical log-branch* of e by

$$L(e) := \sum_{E \rightarrow e} *_E l(E)$$

the *total edge N th-root* by

$$W'_N(e) := \prod_{E \rightarrow e} w'(E)^{*E}$$

and the *total edge charge* by

$$C(e) := \sum_{E \rightarrow e} c(E)$$

where $l(E) = \log(w_k^j) + i f_k^j \pi$, $w(E) = w_k^j$ and $c(E) = c_k^j$ if and only if E is an edge of the abstract tetrahedron (Δ_j, b_j, w^j) , and E is either E_k^j or the opposite edge, and $*_E = *_j$.

Fix an auxiliary order of the, say n , edges of (T, b) . Then we have the analytic maps

$$\begin{aligned} t_L : \mathcal{W}_\infty^s &\rightarrow \mathbb{C}^n, \quad t_L([w; f]) := (L(e_1), \dots, L(e_n)) \\ t_{N, W', c} : \mathcal{W}_\infty^s &\rightarrow \mathbb{C}^n, \quad t_{N, W', c}([w; f]) := (W'_N(e_1), \dots, W'_N(e_n)) \end{aligned}$$

the dependence of $L(e_k)$ (resp. $W'_N(e_k)$) on $[w; f]$ (resp. $[w; f]$ and c) being understood. The following Lemma is evident.

Lemma 3.5. *The restriction of t_L to $G(T, \tilde{b})_\infty$ takes values in $(2i\pi\mathbb{Z})^n$, while the restriction of $t_{N, W', c}$ takes values in $(U_N)^n$, where U_N denotes the set of N th-roots of unity. Both maps are discrete-valued, hence they are constant on the closure of each connected component of the set of non-singular points of $G(T, \tilde{b})_\infty$. Moreover, the restriction of $t_{N, W', c}$ factorizes as*

$$t_{N, W', c} = t'_{N, W} \circ (\text{EXP} \circ l_{N, *, b, c})$$

where $t'_{N, W}$ is a rational regular map defined on $G(T, \tilde{b}, c)_N$ and constant on the irreducible components.

Finally define the analytic subset

$$G_0(T, \tilde{b})_\infty = t_L^{-1}(0, \dots, 0) \subset G(T, \tilde{b})_\infty$$

and, for every N , the algebraic subvariety

$$G_0(T, \tilde{b}, c)_N = \text{EXP} \circ l_{N, *, b, c}(G_0(T, \tilde{b})_\infty) \subset G(T, \tilde{b}, c)_N.$$

As usual, keep the same notation for the restriction of any already defined map. Summing up:

Definition 3.6. The N th rough analytic configuration over (T, \tilde{b}, c) is the family of spaces and maps

$$\mathcal{A}_N^R(T, \tilde{b}, c) := \{\mathcal{W}_\infty^s, \mathcal{W}_N^s, p_\infty, p_N, l_{N, *, b, c}, \mathcal{H}_0, \mathcal{H}_N^e\}.$$

The N th coarse analytic configuration over (T, \tilde{b}, c) is the subfamily of spaces and maps

$$\mathcal{A}_N^C(T, \tilde{b}, c) := \{G_0(T, \tilde{b})_\infty, G_0(T, \tilde{b}, c)_N, p_\infty, p_N, l_{N, *, b, c}, \mathcal{H}_0, \mathcal{H}_N^e\}.$$

We stress that every map $t_{N, W', c}$ is constant on $G_0(T, \tilde{b})_\infty$, and its value depends on the fixed rough charge c , precisely on the edge total charges $(C(e_1), \dots, C(e_n))$.

4. CUSPED MANIFOLDS

Let M be a cusped manifold, and V be as in Section 1.1, that is, a compact oriented connected 3-manifold with one torus boundary component such that M is diffeomorphic to the interior of V . We use the notation \hat{V} as in Section 2.

4.1. The augmented $PSL(2, \mathbb{C})$ -character variety and A -polynomial. Fix a geometric basis (l, m) of the fundamental group $\pi_1(\partial V) \cong \mathbb{Z} \times \mathbb{Z}$. It is given by a couple of oriented simple closed curves on ∂V which meet at one point, transversely and positively. If M is the complement of a hyperbolic knot K in S^3 we can take a canonical longitude l and a meridian m of K .

There is a conjugacy class $[\sigma]$ of representations of $\pi_1(\partial V)$ in $PSL(2, \mathbb{C})$, each one with images isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, acting on the Riemann sphere $\mathbb{CP}^1 = \partial\mathbb{H}^3$ without a common fixed point. The class $[\sigma]$ is obtained as follows: pick two geodesic lines in \mathbb{H}^3 , say γ_l and γ_m , that meet at one point forming a right angle; then define σ by $\sigma(x) = r_x$, where $x = \{l, m\}$ and r_x is the rotation by π around γ_x .

By definition, the $PSL(2, \mathbb{C})$ -character variety of M is the algebro-geometric quotient

$$X'(M) := \text{Hom}'(\pi_1(M), PSL(2, \mathbb{C})) // PSL(2, \mathbb{C})$$

where $PSL(2, \mathbb{C})$ acts by conjugation, and $\text{Hom}'(\pi_1(M), PSL(2, \mathbb{C})) \subset \text{Hom}(\pi_1(M), PSL(2, \mathbb{C}))$ is the subset of representations that do not restrict to an element of $[\sigma]$ on $\pi_1(\partial V)$. Given a peripheral subgroup $\pi_1(\partial V)$ of $\pi_1(M) \cong \pi_1(V)$, denote by

$$R(M) \subset \text{Hom}(\pi_1(V), PSL(2, \mathbb{C})) \times \mathbb{CP}^1$$

the set of couples (r, z) such that z is fixed by $r(\pi_1(\partial V))$. In particular, couples such that $[r]$ restricts to $[\sigma]$ on ∂V are excluded. The *augmented $PSL(2, \mathbb{C})$ -character variety* of M is the algebro-geometric quotient

$$X(M) := R(M) // PSL(2, \mathbb{C})$$

where $PSL(2, \mathbb{C})$ acts by conjugation on $R(M)$ and by Möbius transformations on \mathbb{CP}^1 . Hence, any *augmented character* of M , $\rho = [(\rho, z)] \in X(M)$, is a character of representations $r : \pi_1(M) \rightarrow PSL(2, \mathbb{C})$ together with a choice of fixed points of the peripheral subgroups, invariant by conjugation. The hyperbolic holonomy of M defines a point $\rho_{\text{hyp}} \in X(M)$. In a similar way we can define the $PSL(2, \mathbb{C})$ -character variety $X'(\partial V)$ of ∂V and its augmented version $X(\partial V)$. The inclusion $i : \partial V \rightarrow V$ induces regular maps $i^* : X'(M) \rightarrow X'(\partial V)$ and $i^* : X(M) \rightarrow X(\partial V)$.

Theorem 4.1. (1) ([15, 9]; see [34]) *Both $X'(M)$ and $X(M)$ are complex algebraic affine varieties, and the natural projection $q : X(M) \rightarrow X'(M)$ is a regular map.*
(2) ([21]) ρ_{hyp} (resp. $q(\rho_{\text{hyp}})$) *is a regular point of $X(M)$ (resp. $X'(M)$). Hence it belongs to a unique irreducible component $X_0(M)$ of $X(M)$ (resp. $X'_0(M)$ of $X'(M)$), which is a complex algebraic curve.*
(3) *The restriction of q to $X_0(M)$, $q_0 : X_0(M) \rightarrow X'_0(M)$, is generically $2 : 1$.*
(4) ([16]) *The restricted map $i^* : X'_0(M) \rightarrow X'_0(\partial V)$ (resp. $i^* : X_0(M) \rightarrow X_0(\partial V)$) is generically $1 : 1$. Hence it is a birational isomorphism onto its image.*

For any $\rho \in X(\partial V)$ which is non trivial, let $\bar{\rho}$ be a representative of ρ such that $\bar{\rho}(\pi_1(\partial V))$ fixes $z = \infty \in \mathbb{CP}^1$. For any non zero class $\gamma \in \pi_1(\partial V)$, $\bar{\rho}(\gamma)$ acts on \mathbb{C} as a similarity $w \mapsto \gamma_\rho w + b$, where $\gamma_\rho \in \mathbb{C}^*$ and $b \in \mathbb{C}$. In general γ_ρ is a squared eigenvalue of $\bar{\rho}(\gamma) \in PSL(2, \mathbb{C})$; if $\bar{\rho}(\gamma)$ is loxodromic, then $\gamma_\rho \neq 1$, and the two reciprocally inverse eigenvalues are distinguished by the augmentation, which selects an endpoint, whence an orientation, of γ . Consider the so called *holonomy map*

$$(31) \quad \begin{array}{ccc} \text{hol}_\gamma : & X(\partial V) & \longrightarrow \mathbb{C}^* \\ & \rho & \longmapsto \gamma_\rho. \end{array}$$

By using the above cusp basis (l, m) we get an algebraic isomorphism

$$\text{hol}_m \times \text{hol}_l : X(\partial V) \rightarrow \mathbb{C}^* \times \mathbb{C}^* .$$

Define the rational map

$$\mathfrak{h} : X(M) \rightarrow \mathbb{C}^* \times \mathbb{C}^*, \quad \mathfrak{h} = (\text{hol}_m \times \text{hol}_l) \circ i^* .$$

Following [12], [14], and Dunfield's appendix in [8], consider the plane curve $A(M)$ defined as the closure of the 1-dimensional part of the image $\mathfrak{h}(X(M))$. The (suitably normalized) polynomial generating the ideal of $A(M)$ is by definition the $PSL(2, \mathbb{C})$ *A-polynomial* of M . We denote by $A_0(M)$, and call *geometric component* of $A(M)$, the closure of $\mathfrak{h}(X_0(M))$. It follows from Theorem 4.1 (4) that:

Corollary 4.2. *The restricted map $\mathfrak{h} : X_0(M) \rightarrow A_0(M)$ is a birational isomorphism.*

4.2. Rich components of $G(T)$. Let T be any ideal triangulation of \hat{V} such that the gluing variety $G(T)$ is non empty. There is a natural regular map (see eg. [14] or the appendix of [8])

$$(32) \quad \rho : G(T) \rightarrow X(M) .$$

As $G(T)$ is a complex algebraic curve, $\mathfrak{h}(\rho(G(T)))$ is a union of irreducible components of the plane curve $A(M)$. Assume that $G(T)$ contains a *non-negative* point z_h such that $\rho(z_h) = \rho_{\text{hyp}}$. Recall that “non-negative” means that for every tetrahedron Δ_j of T , the associated cross-ratios z_h^j have non-negative imaginary parts. The point z_h is not necessarily a regular point of the gluing variety, hence in general it could be contained in several irreducible components of $G(T)$.

Definition 4.3. An irreducible component of $G(T)$ is *rich* if it contains z_h and an infinite sequence of points z_n that converge to z_h (in the “strong” topology of $G(T)$ as an analytic space), and correspond to compact closed hyperbolic Dehn fillings of M that geometrically converge to M .

Proposition 4.4. *For every gluing variety $G(T)$ containing a non-negative point z_h with $\rho(z_h) = \rho_{\text{hyp}}$, the set of rich components of $G(T)$ is non empty and finite.*

Proof. This follows from the proof of the hyperbolic Dehn filling Theorem given in [33], and the fact that the algebraic curve $G(T)$ has a finite number of irreducible components. \square

Remark 4.5. In [32], a proof of the hyperbolic Dehn filling Theorem was obtained assuming that the cusped manifold M (allowing several cusps) admits an ideal triangulation T such that $G(T)$ contains a *strictly positive* point z_h representing the hyperbolic holonomy of M . This proof uses also the fact that such a point z_h is a regular point of $G(T)$. By elaborating on the analysis of [32], the paper [13] has provided a clean proof of this regularity result, extended to every strictly positive point of $G(T)$, not necessarily representing the hyperbolic holonomy of M . In [7], the proof of [32] was presented with some mild modifications that allow to avoid the regularity information about z_h . Unfortunately, it did not give yet a complete proof of the hyperbolic Dehn filling Theorem based on gluing varieties, since it is not known if every M admits strictly positive geometric ideal triangulations. The paper [33] filled this gap; it uses any gluing variety $G(T)$ with a non-negative z_h as above, which always exists. In such a situation it is not known that z_h is a regular point, and in fact the proof also uses at some point the arguments of [7]. In [27] it is proved that strictly positive geometric ideal triangulations exist “virtually”, that is, every cusped M has a finite covering \tilde{M} having such an ideal triangulation. On the other hand, passing to \tilde{M} increases the number of cusps.

Proposition 4.6. *Let Z be any rich component of $G(T)$. Then the restricted maps $\rho : Z \rightarrow X_0(M)$ and $\mathfrak{h} \circ \rho : Z \rightarrow A_0(M)$ are generically 1 : 1 and hence are both birational isomorphisms. More precisely, there are maximal non empty Zariski open subsets Ω_Z^0 and Ω_Z of Z , with $\Omega_Z^0 \subset \Omega_Z$, such that:*

- (1) $\rho(\Omega_Z)$ and $\rho(\Omega_Z^0)$ are Zariski open subsets of $X_0(M)$ (resp. $\mathfrak{h} \circ \rho(\Omega_Z)$ and $\mathfrak{h} \circ \rho(\Omega_Z^0)$ are Zariski open subsets of $A_0(M)$).
- (2) The restriction of ρ and $\mathfrak{h} \circ \rho$ to Ω_Z is 1 : 1 onto its image, and $z_h \in \Omega_Z$. The restriction of ρ and $\mathfrak{h} \circ \rho$ to Ω_Z^0 is a regular rational isomorphism onto its image.

Proof. This goes step by step as the proof of Theorem 3.1 in [16]. The main ingredients are the Gromov-Thurston “volume rigidity” for closed hyperbolic manifolds, the fact that the volume of representations yields a well defined function on the normalization of the smooth projective model of $A_0(M)$, and the existence on $X_0(M)$ of infinite sequences of points corresponding to the holonomies of compact closed hyperbolic Dehn fillings of M that converge geometrically to M . This last key fact is just ensured by the definition of a rich component, and the volume of representations just lifts to the function (9) on $G(T)$. \square

If T is endowed with a weak branching (T, \tilde{b}) , then the above discussion can be rephrased in terms of the gluing variety $G(T, \tilde{b})$, accordingly with Section 2. Recall that a system of cross ratios w^j on the branched tetrahedra (Δ_j, b_j) of (T, \tilde{b}) is non negative if $z^j = (w^j)^{*j}$ is non negative.

4.3. Refined analytic configurations for one-cusped manifolds. Let (T, \tilde{b}) be a weakly branched ideal triangulation of \hat{V} .

Definition 4.7. A *global charge* c on (T, \tilde{b}) is a rough global charge satisfying the following additional global constraint: *For every edge e of T , the total edge charge $C(e)$ is constant and equal to 2.*

Note that with this specialization of the charges, the defining equations of the algebraic variety $G_0(T, b, c)_N$ in the coarse analytic configurations are for every edge e of T of the form

$$(33) \quad W'(e) = \zeta_N^{-1}.$$

Note that $\zeta_N^{-1} = \zeta_N^{2m}$. We have:

Proposition 4.8. (1) *Every global charge c on (T, \tilde{b}) determines a c -weight (h_c, k_c) of V .*
 (2) *For every c -weight (h_c, k_c) of V and every ideal triangulation (T, \tilde{b}) of \hat{V} , there exists a global charge c on (T, \tilde{b}) with c -weight equal to (h_c, k_c) .*

Proposition 4.9. *Assume that the gluing variety $G(T, \tilde{b})$ is non empty, and let $\rho : G(T, \tilde{b}) \rightarrow X(M)$ be as in (32).*

- (1) *Every point $[w; f] \in G_0(T, \tilde{b})_\infty$ determines an f -weight (h_f, k_f) of $(V, \rho(w))$.*
- (2) *For every point $\rho \in \rho(G(T, \tilde{b}))$ and every f -weight (h_f, k_f) of (V, ρ) , there exists a point $[w; f]$ of $G_0(T, \tilde{b})_\infty$ with f -weight equal to (h_f, k_f) .*

Proposition 4.8 and 4.9 are essentially reformulations of results of Neumann [30, 31] already used in [1, 2, 3]. We recall below the constructions underlying the statements (1), which claim the existence of the pairs $(\gamma(c), \gamma_2(c))$ and $(L([w; f]), \gamma_2(f))$ in Definition 4.10. The statements (2) are much more subtle; they rely on the material developed in the proof of Theorem 1.1.

Denote by T_0 the cell complex obtained from T by removing an open cone neighborhood of its vertex. Hence T_0 is the result of gluing tetrahedra with truncated vertices, and the polyhedron underlying T_0 is homeomorphic to V . Denote by ∂T_0 the induced triangulation of ∂V . Represent any non zero class in $H_1(\partial V; \mathbb{Z})$ by *normal loops*, that is, a disjoint union of oriented essential simple closed curves in ∂V , transverse to the edges of ∂T_0 and such that no curve enters and exits a 2-simplex by a same edge. The intersection of a normal loop, say C , with a 2-simplex F consists of a disjoint union of arcs, each of which turns around a vertex of F ; if F is a cusp section of the tetrahedron Δ of T , for every vertex v of F we denote by E_v the edge of Δ containing v , and by $*_v$ the branching sign of Δ . We write $C \rightarrow E_v$ to mean that some subarcs of C turn around v . We count them algebraically, by using the orientation of C : if there are s_+ (resp. s_-) such subarcs whose orientation is compatible with (resp. opposite to) the orientation of ∂V as viewed from v , then we set $\text{ind}(C, v) := s_+ - s_-$. For every point $[w; f] \in G_0(T, \tilde{b})_\infty$ and every global charge c on (T, \tilde{b}) , one defines cohomology classes $L([w; f]) \in H^1(\partial V; \mathbb{C})$ and $\gamma(c) \in H^1(\partial V; \mathbb{Z})$ by evaluating on normal loops C in ∂V via the following formulas:

$$(34) \quad L([w; f])([C]) := \sum_{C \rightarrow E_v} *_v \text{ind}(C, v) l(E_v) = \sum_{C \rightarrow E_v} *_v \text{ind}(C, v) (\log(w(E_v)) + \pi \sqrt{-1} f(E_v))$$

$$(35) \quad \gamma(c)([C]) := \sum_{C \rightarrow E_v} \text{ind}(C, v) c(E_v) .$$

Additional classes $\gamma_2(c), \gamma_2(f) \in H^1(V; \mathbb{Z}/2\mathbb{Z})$ are defined similarly, by using normal loops in T and taking the sum mod(2) of the charges or flattenings, respectively, of the edges we face along the loops. The reduction mod(2) of $\gamma(c)$ coincides with the image of $\gamma_2(c)$ under the map induced on cohomology by the inclusion $i : \partial V \rightarrow V$. Hence $r(\gamma(c)) = i^*(\gamma_2(c))$. Also, denoting by $d_w \in H^1(\partial V; \mathbb{C}/2i\pi)$ the log of the dilation part of the restriction of $\rho(w)$ to $\pi_1(\partial V)$, for all $a \in H_1(\partial V; \mathbb{Z})$ we have

$$(36) \quad L([w; f])(a) = d_w(a) \bmod(i\pi) , \quad (L([w; f])(a) - d_w(a))/i\pi = i^*(\gamma_2(f))(a) \bmod(2) .$$

This proves consistency of the following definition with the defining constraints of weights, see (1) and (2) in the Introduction.

Definition 4.10. The f -weight (k_f, h_f) of a point $[w; f] \in G_0(T, \tilde{b})_\infty$, and the c -weight (k_c, h_c) of a global charge c on (T, \tilde{b}) , are defined respectively as the pairs of cohomology classes

$$(L([w; f]), \gamma_2(f)) \in H^1(\partial V; \mathbb{C}) \times H^1(V; \mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad (\gamma(c), \gamma_2(c)) \in H^1(\partial V; \mathbb{Z}) \times H^1(V; \mathbb{Z}/2\mathbb{Z}).$$

Example 4.11. ((T_1, \tilde{b}) continued). Consider the weakly branched triangulation (T_1, \tilde{b}) of M_1 as in Example 2.10. Every point w of the gluing variety $G(T_1, \tilde{b})$ gives rise to a class

$$\exp(d_w) \in H^1(\partial V; \mathbb{C}^*) .$$

According to [17], for a suitable choice of geometric basis (λ, μ) of $H_1(\partial V; \mathbb{Z})$ we have

$$\exp(d_w)(\lambda) = W_1/w_1, \quad \exp(d_w)(\mu) = (W_1/W_2)^2 .$$

By taking the log we get the class $d_w \in H^1(\partial V; \mathbb{C}/2i\pi)$ given by

$$d_w(\lambda) = \log(W_1) - \log(w_1), \quad d_w(\mu) = 2(\log(W_1) - \log(W_2)) .$$

For simplicity let us deal with the positive points in the (unique) rich component of the gluing variety $G(T_1, \tilde{b})$ which contains the hyperbolic solution, $w = W = \exp(i\pi/3)$. Let us denote by f_j (resp. F_j) the flattening variables at the top (resp. bottom) crossing of Figure 5. For every such a positive point, the *flattening equations* are

$$(37) \quad f_0 + f_1 + f_2 = F_0 + F_1 + F_2 = -1, \quad f_0 + 2f_1 + F_0 + 2F_1 = -2, \quad f_0 + 2f_2 + F_0 + 2F_2 = -2 .$$

In fact, for an arbitrary point w in the gluing variety, the flattening equations are derived from the equations for the log-branches $l_j := \log(w_j) + i\pi f_j$ and $L_j := \log(W_j) + i\pi F_j$. These are (see (11))

$$(38) \quad l_0 + l_1 + l_2 = L_0 + L_1 + L_2 = 0, \quad l_0 + 2l_1 + L_0 + 2L_1 = 0, \quad l_0 + 2l_2 + L_0 + 2L_2 = 0.$$

Now, since w is positive, the pairs of first two equations in the two systems (37) and (38) are clearly equivalent to each other. Although a bit subtler, this is true also for the remaining sets of equations, because the consistency relations (11) imply that the sum of the arguments of the cross ratios around both edges is *exactly* equal to 2π (see [7], Lemma E.6.1).

The *charge equations* are formally obtained from the flattening ones by replacing in (37) each f_j with $-c_j$ and each F_j with $-C_j$. Solving the systems we get

$$f_0 = -(2f_1 + F_1 - F_2 + 1), \quad F_0 = -(F_1 + F_2 + 1), \quad f_2 = f_1 + F_1 - F_2$$

$$c_0 = -(2c_1 + C_1 - C_2 - 1), \quad C_0 = -(C_1 + C_2 - 1), \quad c_2 = c_1 + C_1 - C_2$$

$$l_0 = -(2l_1 + L_1 - L_2), \quad L_0 = -(L_1 + L_2), \quad l_2 = l_1 + L_1 - L_2.$$

Let us turn to the *boundary c-weight*

$$k_c \in H^1(\partial V; \mathbb{Z})$$

and, for every positive w as above, to the *boundary f-weight*

$$k_f = k_f(w) \in H^1(\partial V; \mathbb{C}).$$

They are given by

$$k_c(\lambda) = C_1 - c_1, \quad k_c(\mu) = 2C_1 - 2C_2$$

$$k_f(\lambda) = L_1 - l_1, \quad k_c(\mu) = 2L_1 - 2L_2.$$

At the complete solution we get

$$k_f(\lambda) = F_1 - f_1, \quad k_f(\mu) = 2F_1 - 2F_2.$$

Hence $k_f \in H^1(\partial V; \mathbb{Z})$, and moreover $k_f(\mu) = 0 \pmod{2}$. In this case the flattening solutions can be rewritten as

$$f_0 = -(k_f(\mu)/2 + 2f_1 + 1), \quad f_2 = f_1 + k_f(\mu)/2$$

$$F_0 = -(2k_f(\lambda) - k_f(\mu)/2 + 2f_1 + 1), \quad F_1 = k_f(\lambda) + f_1, \quad F_2 = f_1 + k_f(\lambda) - k_f(\mu)/2.$$

Hence for any given boundary f -weight k_f , there is a family of flattenings that realize it depending on the unique free parameter f_1 . For a general positive w , we have the log-branch relations

$$l_0 = -(k_f(\mu)/2 + 2l_1), \quad l_2 = l_1 + k_f(\mu)/2$$

$$L_0 = -(2k_f(\lambda) - k_f(\mu)/2 + 2l_1), \quad L_1 = k_f(\lambda) + l_1, \quad L_2 = l_1 + k_f(\lambda) - k_f(\mu)/2.$$

Similarly for the charges we have

$$c_0 = -(k_c(\mu)/2 + 2c_1 - 1), \quad c_2 = c_1 + k_c(\mu)/2$$

$$C_0 = -(2k_c(\lambda) - k_c(\mu)/2 + 2c_1 - 1), \quad C_1 = k_c(\lambda) + c_1, \quad C_2 = c_1 + k_c(\lambda) - k_f(\mu)/2$$

where the free parameter is now c_1 . By taking $f_1 = c_1 = 0$, the relevant pairs of parameters that enter the formulas of the tensors \mathcal{R}_N^e are

$$(f_0, f_1) = (-1 - k_f(\mu)/2, 0), \quad (c_0, c_1) = (1 - k_c(\mu)/2, 0)$$

$$(F_0, F_1) = (-1 - 2k_f(\lambda) + k_f(\mu)/2, k_f(\lambda))$$

$$(C_0, C_1) = (1 - 2k_c(\lambda) + k_c(\mu)/2, k_c(\lambda))$$

stressing in this way a pure dependence on the weights. Finally, the enhanced QH state sums have the form

$$\mathcal{H}_N^e(T_1, \tilde{b}, w, W, f, F, c, C) = \sum_{i,j,l,k,I,J=0}^{N-1} \mathcal{R}_N(1, w, f, c)_{k,i}^{i,j} \mathcal{R}_N(1, W, F, C)_{j,i}^{I,J} (\mathcal{Q})_I^l (\mathcal{Q}^2)_J^k$$

where we have used the periodicity mod(3) of the matrix Q , and we have quoted that $*_b = 1$ for both tetrahedra. \square

Let now Z be a rich component of $G(T, \tilde{b})$. Denote $Z_\infty := p_\infty^{-1}(Z)$ (an analytic subspace of $G_0(T, \tilde{b})_\infty$), and let $Z_{\infty,0}$ be the union of connected components of Z_∞ made of points $[w; f]$ such that

$$(39) \quad i^*(\gamma_2(f)) = 0 \in H^1(\partial V; \mathbb{Z}/2\mathbb{Z}) .$$

Recall the basis (l, m) of $\pi_1(\partial V)$. Because of (36), for every point $[w; f] \in Z_{\infty,0}$ we have:

- the equality

$$(40) \quad (e^{L([w;f])}(l), e^{L([w;f])}(m)) = \mathfrak{h} \circ \rho(w)$$

- for every global charge c on (T, \tilde{b}) , the formula (with C a normal loop in ∂V)

$$(41) \quad \begin{aligned} L_{N,c}([w; f])(C) &:= \sum_{C \rightarrow E_v} *_v \text{ind}(C, v) l_{N, *_v, c}(E_v) \\ &= \sum_{C \rightarrow E_v} \frac{*_v}{N} \text{ind}(C, v) (\log(w(E_v)) + \pi\sqrt{-1}(N+1)(f(E_v) - *_v c(E_v))) \end{aligned}$$

$$(42) \quad = \frac{1}{N} (L([w; f])([C]) - \pi\sqrt{-1}(N+1)\gamma(c)([C]) + \pi\sqrt{-1} \underbrace{\sum_{C \rightarrow E_v} *_v \text{ind}(C, v) f(E_v)}_{\in 2\mathbb{Z}})$$

yields a cohomology class $L_{N,c}([w; f]) \in H^1(\partial V; \mathbb{C}/2\pi i)$ such that

$$(43) \quad ((e^{L_{N,c}([w;f])}(l)})^N, (e^{L_{N,c}([w;f])}(m)})^N) = \mathfrak{h} \circ \rho(w) .$$

If we allow $[w; f]$ to live in $Z_\infty \setminus Z_{\infty,0}$, the sum in (42) can be odd, and the equalities (40) & (43) hold true only up to signs in each component. More precisely, for all $[w; f] \in Z_\infty$ the sum in (42) is an integer expression of the residue class $\gamma_2(f)([C]) \in \mathbb{Z}/2\mathbb{Z}$, so that we have a well-defined cohomology class

$$(44) \quad L_{N,c}([w; f]) \in H^1(\partial V; \mathbb{C}/2\pi i)$$

such that

$$L_{N,c}([w; f]) = \frac{1}{N} (L([w; f]) - \pi i(N+1)\gamma(c) + i\pi i^*(\gamma_2(f))) \pmod{2\pi i}$$

where, as usual, $i^* : H^1(V; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(\partial V; \mathbb{Z}/2\mathbb{Z})$ is induced by the inclusion map $i : \partial V \rightarrow V$. The spaces $Z_{\infty,0}$ and $A_0(M)$ are related as follows (see Proposition 4.6). Let

$$A_0(M)_\infty := \pi_\infty^{-1}(A_0(M)) , \quad A_0(M)_N := \text{EXP} \circ l_{N, k_c}(A_0(M)_\infty)$$

where

$$(45) \quad \pi_\infty : \mathbb{C}^2 \rightarrow (\mathbb{C}^*)^2 , \quad ([\lambda; p], [\lambda; q]) \mapsto (\lambda, \mu)$$

is the universal covering map (we identify \mathbb{C} with the Riemann surface of log, so that $[\lambda; p]$ stands for $\log(\lambda) + 2\pi\sqrt{-1}p$, with $p \in \mathbb{Z}$ and identifications as in the first row of (15), with q removed), and for every integral class $k_c \in H^1(\partial V; \mathbb{Z})$, we define the map $l_{N, k_c} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by

$$(46) \quad l_{N, k_c}([\lambda; p], [\mu; q]) := \left(\frac{1}{N} (\log(\lambda) + 2\pi\sqrt{-1}p - \pi\sqrt{-1}(N+1)k_c(l)), \right. \\ \left. \frac{1}{N} (\log(\mu) + 2\pi\sqrt{-1}q - \pi\sqrt{-1}(N+1)k_c(m)) \right) .$$

The restricted map $\pi_\infty : A_0(M)_\infty \rightarrow A_0(M)$ and the map $\pi_N : A_0(M)_N \rightarrow A_0(M)$, $(u, v) \mapsto (u^N, v^N)$, define respectively a $\mathbb{Z} \times \mathbb{Z}$ - and a $\mathbb{Z}/N \times \mathbb{Z}/N$ -covering of $A_0(M)$.

Definition 4.12. The *classical* and *quantum log-holonomies* are the maps defined on $Z_{\infty,0}$ by the components of the classes $L([w; f]) \in H^1(\partial V; \mathbb{C})$ and $\exp \circ L_{N,c}([w; f]) \in H^1(\partial V; \mathbb{C}^*)$, $[w; f] \in Z_{\infty,0}$, in the basis (l, m) of $\pi_1(\partial V)$:

$$\begin{aligned} \text{loghol}_{\infty} : Z_{\infty,0} &\longrightarrow A_0(M)_{\infty} \\ [w; f] &\longmapsto (L([w; f])(l), L([w; f])(m)) . \\ \text{loghol}_{N,c} : Z_{\infty,0} &\longrightarrow A_0(M)_N \\ [w; f] &\longmapsto (e^{L_{N,c}([w; f])(l)}, e^{L_{N,c}([w; f])(m)}) . \end{aligned}$$

Note that (40) & (43) imply that loghol_{∞} (resp. $\text{loghol}_{N,c}$) maps into $A_0(M)_{\infty}$ (resp. $A_0(M)_N$). The identification (17) of \mathcal{W}_{∞}^s as a product of Riemann surfaces, and the fact that $Z_{\infty} \subset G_0(T, \tilde{b})_{\infty}$ is an analytic subspace, imply that both loghol_{∞} and $\text{loghol}_{N,c}$ are analytic maps.

Remark 4.13. By removing the assumption (39), the maps loghol_{∞} and $\text{loghol}_{N,c}$ extend to the whole of Z_{∞} . Under this extension the target space of loghol_{∞} becomes the union of the four connected covering spaces $A_0(M)_{\infty}^{\varepsilon, \varepsilon'} := (\pi_{\infty}^{\varepsilon, \varepsilon'})^{-1}(A_0(M))$, where $\pi_{\infty}^{\varepsilon, \varepsilon'} : \mathbb{C}^2 \rightarrow (\mathbb{C}^*)^2$, $(\lambda, \mu) \mapsto (\varepsilon e^{\lambda}, \varepsilon' e^{\mu})$, $\varepsilon, \varepsilon' \in \{0, 1\}$. The target space of $\text{loghol}_{N,c}$ becomes the union of the spaces $\text{EXP} \circ l_{N, k_c}(A_0(M)_{\infty}^{\varepsilon, \varepsilon'})$. The component $A_0(M)_{\infty} = A_0(M)_{\infty}^{0,0}$ is a natural one: we explain in Section 4.4 its relation with the Chern-Simons line bundle over ∂V .

We deduce immediately from the formulas (34), (41) and (46):

Lemma 4.14. *For every N and every global charge c on (T, \tilde{b}) we have a commutative diagram:*

$$\begin{array}{ccccc} A_0(M) & \xleftarrow{\pi_N} & A_0(M)_N & & \\ & \nwarrow \eta \circ \rho & \uparrow & \nwarrow \text{loghol}_{N,c} & \\ & & Z & \xleftarrow[p_{\infty}]{e^{l_{N, k_c}}} & Z_{\infty,0} \\ & \swarrow \eta \circ \rho & & \swarrow \text{loghol}_{\infty} & \\ A_0(M) & \xleftarrow{\pi_{\infty}} & A_0(M)_{\infty} & & \end{array}$$

where l_{N, k_c} is defined by using the boundary weight k_c of c , and we denote $\text{EXP} \circ l_{N, k_c}$ by $e^{l_{N, k_c}}$.

Let now \mathcal{F} be any finite family of rich components, possibly contained in different gluing varieties associated to different weakly branched triangulations of \hat{V} . Recall Proposition 4.6. Define the following non empty Zariski open subsets of $X_0(M)$:

$$\Omega_{\mathcal{F}} = \cap_{Z \in \mathcal{F}} \rho(\Omega_Z) , \quad \Omega_{\mathcal{F}}^0 = \cap_{Z \in \mathcal{F}} \rho(\Omega_Z^0) .$$

If $Z \in \mathcal{F}$ is a rich component of $G(T, \tilde{b})$, define

$$\Omega_{\mathcal{F}}(Z) := \rho|_Z^{-1}(\Omega_{\mathcal{F}}) , \quad \Omega_{\mathcal{F}}^0(Z) := \rho|_Z^{-1}(\Omega_{\mathcal{F}}^0) , \quad \Omega_{\mathcal{F}}(Z)_{\infty,0} := p_{\infty}^{-1}(\Omega_{\mathcal{F}}(Z)) \cap Z_{\infty,0} .$$

Clearly we have:

- $\Omega_{\mathcal{F}}^0 \subset \Omega_{\mathcal{F}}$.
- The restriction of ρ to $\Omega_{\mathcal{F}}(Z)$ is a rational regular map which is 1 : 1 onto $\Omega_{\mathcal{F}}$. Moreover, the non negative point $w_h^Z \in Z$ such that $\rho(w_h^Z) = \rho_{\text{hyp}}$ belongs to $\Omega_{\mathcal{F}}(Z)$.
- For every component $Z \in \mathcal{F}$, the restriction of ρ to $\Omega_{\mathcal{F}}^0(Z)$ is a rational regular isomorphism onto $\Omega_{\mathcal{F}}^0$.

The following Lemma is an immediate consequence of Proposition 4.9.

Lemma 4.15. *Let \mathcal{F} be any finite family of rich components of gluing varieties associated to weakly branched ideal triangulations of \hat{V} . For every component Z in \mathcal{F} , every point $\rho \in \Omega_{\mathcal{F}}$, and every f -weight (k_f, h_f) of (V, ρ) such that $i^*(h_f) = 0$, there is a point $[w; f]_Z \in \Omega_{\mathcal{F}}(Z)_{\infty,0}$ such that $(L([w; f]_Z), \gamma_2(f)) = (k_f, h_f)$.*

Put

$$A_{0,\mathcal{F}}(M) := \mathfrak{h}(\Omega_{\mathcal{F}}) , \quad A_{0,\mathcal{F}}(M)_{\infty} := \pi_{\infty}^{-1}(A_{0,\mathcal{F}}(M)) , \quad A_{0,\mathcal{F}}(M)_N := \text{EXP} \circ l_{N,k_c}(A_{0,\mathcal{F}}(M)_{\infty}) .$$

By replacing $G(T, \tilde{b})$ with $\Omega_{\mathcal{F}}(Z) \subset Z \subset G(T, \tilde{b})$, and $G_0(T, \tilde{b})_{\infty}$ with $\Omega_{\mathcal{F}}(Z)_{\infty,0} \subset Z_{\infty,0} \subset G_0(T, \tilde{b})_{\infty}$, and taking the restrictions of each of the spaces and maps that form the sequence of coarse analytic configurations $\{\mathcal{A}_N^C(T, \tilde{b}, c)\}$, one obtains a sequence of sub-configurations

$$\mathcal{A}_N(c, Z, \mathcal{F}) := \{\Omega_{\mathcal{F}}(Z)_{\infty,0}, \Omega_{\mathcal{F}}(Z)_N, p_{\infty}, p_N, l_{N,*_b,c}, \mathcal{H}_0, \mathcal{H}_N^e\} .$$

By Lemma 4.14, for all N the sub-configuration $\mathcal{A}_N(c, Z, \mathcal{F})$ gives rise to a commutative diagram:

$$\begin{array}{ccccc}
 & & \Omega_{\mathcal{F}}(Z)_N & & \\
 & & \uparrow e^{l_{N,*_b,c}} & \searrow \mathcal{H}_N^{e'} & \\
 A_{0,\mathcal{F}}(M) & \xleftarrow{\pi_N} & A_{0,\mathcal{F}}(M)_N & \xleftarrow{\loghol_{N,c}} & \mathbb{C}/\mu_{4N} \\
 & \searrow \mathfrak{h} \circ \rho & \downarrow \wedge & \nearrow \mathcal{H}_N^e & \\
 & \Omega_{\mathcal{F}}(Z) & \xleftarrow{p_{\infty}} & \Omega_{\mathcal{F}}(Z)_{\infty,0} & \\
 & \searrow \mathfrak{h} \circ \rho & \downarrow e^{l_{N,k_c}} & \nearrow \loghol_{\infty} & \\
 A_{0,\mathcal{F}}(M) & \xleftarrow{\pi_{\infty}} & A_{0,\mathcal{F}}(M)_{\infty} & \xrightarrow{\mathcal{H}_0} & \mathbb{C}
 \end{array}$$

The following two results show that if we fix the bulk weight h_f together with the c -weight (h_c, k_c) , then \mathcal{H}_0 and \mathcal{H}_N^e factor through \loghol_{∞} and $\loghol_{N,c}$, and hence induce maps on $A_{0,\mathcal{F}}(M)_{\infty}$ and $A_{0,\mathcal{F}}(M)_N$, respectively.

Theorem 4.16. *Let \mathcal{F} be any finite family of rich components Z of gluing varieties $G(T_Z, \tilde{b}_Z)$ associated to weakly branched ideal triangulations (T_Z, \tilde{b}_Z) of \hat{V} .*

(1) *Assume that all weak branchings \tilde{b}_Z are genuine branchings b_Z . Let $\rho \in \Omega_{\mathcal{F}}$, the f -weight (h_f, k_f) and $[w; f]_Z \in \Omega_{\mathcal{F}}(Z)_{\infty,0}$ be as in Lemma 4.15, and (h_c, k_c) be any c -weight represented on the triangulation T_Z be a global charge c_Z . Denote $\mathcal{P} = (V, \rho, h_f, k_f, h_c, k_c)$. Then the scalar:*

- $\mathcal{H}_0^{\mathcal{F}}(\mathcal{P}) := \mathcal{H}_0(T_Z, b_Z)([w; f]_Z)$ *is a well defined invariant of \mathcal{P} and does not depend on the c -weight (h_c, k_c) ;*
- $\mathcal{H}_N^{\mathcal{F}}(\mathcal{P}) := \mathcal{H}_N(T_Z, b_Z, c_Z)([w; f]_Z)$, $N \geq 3$ *odd, is a well defined invariant of \mathcal{P} up to multiplication by $2N$ th roots of unity, and depends on k_f through its reduction mod(N) only.*

(2) *If \tilde{b}_Z is only a weak branching, then the same results hold true, up to multiplication by 6th roots of unity in the case of \mathcal{H}_0 , and up to multiplication by 4Nth roots of unity by replacing \mathcal{H}_N with the enhanced state sums \mathcal{H}_N^e of Section 3.*

Proof. Statement (1) means that, up to the phase ambiguity, the scalars do not depend on the arbitrary choices of the component Z (and hence also of the branched triangulation (T_Z, b_Z)), the charge c_Z , or the point $[w; f]_Z$ that we have made in order to encode \mathcal{P} . The proof goes by applying the whole technology of *QH transits*, that we have developed in [1, 2, 3] to construct the QH invariants. Here we limit ourselves to recall that given two realizations of the pattern \mathcal{P} , say:

- $[w; f]_Z \in \Omega_{\mathcal{F}}(Z)_{\infty,0}$, included into some sub-configuration $\mathcal{A}_N(c_Z, Z, \mathcal{F})$, and
- $[w'; f']_{Z'} \in \Omega_{\mathcal{F}}(Z')_{\infty,0}$, included into some possibly different sub-configuration $\mathcal{A}_N(c_{Z'}, Z', \mathcal{F})$,

then there exists a finite sequence of QH transits (based on triangulation moves) from $[w; f]_Z$ to a point $[w''; f'']_{Z'} \in \Omega_{\mathcal{F}}(Z')_{\infty,0}$, included into a sub-configuration $\mathcal{A}_N(c'_{Z'}, Z', \mathcal{F})$ (the same as for $[w'; f']_{Z'}$ except for the global charge $c'_{Z'}$, which may be different from $c_{Z'}$), encoding the same pattern \mathcal{P} , and such that

$$\mathcal{H}_N(T_Z, b_Z, c_Z)([w; f]_Z) \equiv_{2N} \mathcal{H}_N(T_{Z'}, b_{Z'}, c'_{Z'})([w''; f'']_{Z'}) .$$

It follows from the very definition of the Zariski open set $\Omega_{\mathcal{F}} \subset X_0(M)$ that necessarily w'' and w' are equal, that is, $[w''; f'']_{Z'} = [w'; f'']_{Z'}$. A further transit argument shows finally that

$$\mathcal{H}_N(T_{Z'}, b_{Z'}, c'_{Z'})([w'; f'']_{Z'}) \equiv_{2N} \mathcal{H}_N(T_{Z'}, b_{Z'}, c_{Z'})([w'; f']_{Z'}) .$$

Since the QH state sums (27) depend only on the reduction mod(N) of flattenings, the dependence of $\mathcal{H}_N^{\mathcal{F}}$, $N \geq 3$, on $k_f \bmod(N)$ follows immediately. The same QH transits argument apply to $\mathcal{H}_0^{\mathcal{F}}$, thus proving (1).

In order to prove the statement (2) we can adopt a similar strategy, by using either Section 4.3 of [2] (in particular Theorem 4.4 and Remark 4.5) in the case of \mathcal{H}_0 , or the results of Section 6 (in particular about the behaviour of the enhanced QH state sums under a change of weak branching). \square

The above constructions depend on the choice of the family \mathcal{F} . In order to make a canonical choice we can take the family \mathcal{F}_{EP} of rich components Z contained in any weakly branched EP-triangulation of \hat{V} (recall Proposition 2.2). By definition, the analytic configuration $\mathcal{A}_N(Y)$, $Y = (V, (h_c, k_c))$, is the family of analytic sub-configurations $\mathcal{A}_N(c, Z, \mathcal{F}_{\text{EP}})$ for varying charges and rich components $Z \in \mathcal{F}_{\text{EP}}$.

Corollary 4.17. *The invariants $\mathcal{H}_0^{\mathcal{F}_{\text{EP}}}(\mathcal{P})$ and $\mathcal{H}_N^{\mathcal{F}_{\text{EP}}}(\mathcal{P})$ of patterns $\mathcal{P} = (V, \rho, h_f, k_f, h_c, k_c)$ based on M , with $\rho \in \Omega_{\mathcal{F}_{\text{EP}}}$, fixed c -weight $\omega_c := (h_c, k_c)$, and fixed bulk f -weight $h_f \in H^1(V; \mathbb{Z}/2\mathbb{Z})$ such that $i^*(h_f) = 0$, define respectively an analytic function*

$$\mathcal{H}_0^{\mathcal{F}_{\text{EP}}, h_f} : A_{0, \mathcal{F}_{\text{EP}}}(M)_{\infty} \rightarrow \mathbb{C}$$

and a regular rational function

$$(\mathcal{H}_N^e)^{\mathcal{F}_{\text{EP}}, h_f, \omega_c} : A_{0, \mathcal{F}_{\text{EP}}}(M)_N \rightarrow \mathbb{C}/\mu_{4N}$$

that give rise to a commutative diagram

$$\begin{array}{ccccc}
 & & \Omega_{\mathcal{F}_{\text{EP}}}(Z)_N & & \\
 & & \uparrow \mathcal{H}_N^e & & \\
 A_{0, \mathcal{F}_{\text{EP}}}(M) & \xleftarrow{\pi_N} & A_{0, \mathcal{F}_{\text{EP}}}(M)_N & \xrightarrow{\mathcal{H}_N^{\mathcal{F}_{\text{EP}}, h_f, \omega_c}} & \mathbb{C}/\mu_{4N} \\
 & \nwarrow \mathfrak{h} \circ \rho_Z & \uparrow \text{loghol}_{N, c} & \searrow e^{I_N, *_{b, c}} & \\
 & \Omega_{\mathcal{F}_{\text{EP}}}(Z) & \xleftarrow{p_{\infty}} & \Omega_{\mathcal{F}_{\text{EP}}}(Z)_{\infty, 0} & \\
 & \nwarrow \mathfrak{h} \circ \rho_Z & \uparrow e^{I_N, k_c} & \searrow \text{loghol}_{\infty} & \\
 A_{0, \mathcal{F}_{\text{EP}}}(M) & \xleftarrow{\pi_{\infty}} & A_{0, \mathcal{F}_{\text{EP}}}(M)_{\infty} & \xrightarrow{\mathcal{H}_0^{\mathcal{F}_{\text{EP}}, h_f}} & \mathbb{C}
 \end{array}$$

Proof. Any point x in the image of loghol_{∞} or $\text{loghol}_{N, c}$ determines a unique point $\rho = \mathfrak{h}^{-1}(\pi_{\infty}(x))$ or $\mathfrak{h}^{-1}(\pi_N(x))$ in $\Omega_{\mathcal{F}_{\text{EP}}}$. If $x \in \text{Image}(\text{loghol}_{\infty})$, by Definition 4.10 it determines also a boundary f -weight k_f , and by Lemma 4.15 any boundary f -weight k_f for ρ such that $(k_f - d_{\rho})/i\pi$ is the zero class in $H^1(\partial V; \mathbb{Z}/2\mathbb{Z})$ is realized by a point of $\pi_{\infty}^{-1}(\mathfrak{h}(\rho))$ in the image of loghol_{∞} (see (36)). Hence the tuples (x, h_f) and (ρ, h_f, k_f) with $i^*(h_f) = 0$ are in one-to-one correspondence. If $x \in \text{Image}(\text{loghol}_{N, c})$ and a boundary c -weight k_c is given, x determines the reduction mod(N) of k_f . Hence the tuples (x, h_f, h_c, k_c) and $(\rho, h_f, k_f \bmod(N), h_c, k_c)$ with $i^*(h_f) = 0$ are in one-to-one correspondence. From this and Theorem 4.16 it follows that \mathcal{H}_0 and \mathcal{H}_N^e descend to maps $\mathcal{H}_0^{\mathcal{F}_{\text{EP}}, h_f}$ and $(\mathcal{H}_N^e)^{\mathcal{F}_{\text{EP}}, h_f, \omega_c}$ defined on the image of loghol_{∞} and $\text{loghol}_{N, c}$ respectively. Since \mathcal{H}_0 is analytic and π_{∞} and p_{∞} are covering maps, $\mathcal{H}_0^{\mathcal{F}_{\text{EP}}, h_f}$ is analytic too and it extends uniquely to an analytic function on $A_{0, \mathcal{F}_{\text{EP}}}(M)_{\infty}$. Similarly one shows that $(\mathcal{H}_N^e)^{\mathcal{F}_{\text{EP}}, h_f, \omega_c}$ can be extended in a unique way to a regular rational function on $A_{0, \mathcal{F}_{\text{EP}}}(M)_N$. \square

Proof of Theorem 1.1. For the first two statements, apply Theorem 4.16 (2) to the family $\mathcal{F} = \mathcal{F}_{\text{EP}}$, and set $\Omega(M) := \Omega_{\mathcal{F}_{\text{EP}}}$. The third statement follows from Corollary 4.17 by pulling back the covering π_N via \mathfrak{h} . As explained in Remark 4.13, the assumption $i^*(h_f) = 0$ in these statements is not a necessary one; if removed one gets more generally functions

$$A_{0, \mathcal{F}_{\text{EP}}}(M)_N^{\varepsilon, \varepsilon'} \rightarrow \mathbb{C}/\mu_{4N}$$

where $\varepsilon, \varepsilon' \in \{\pm 1\}$.

For the very gentle version of Theorem 1.1 mentioned in Remark 1.8, instead of \mathcal{F}_{EP} use the family $\mathcal{F}_{\text{EP}}^b$ of rich components Z contained in any genuinely branched EP-triangulation of \hat{V} . By definition M is very gentle if $\mathcal{F}_{\text{EP}}^b \neq \emptyset$. Then apply the statement (1) of Theorem 4.16.

Finally, we prove point (4) of the theorem. Let $\mathcal{T} = (T, \tilde{b}, w, f, c)$ be a QH triangulation of the pattern $(V, \rho, (h, k))$, and $(\Delta_j, b^j, w^j, f^j, c^j)$, $j \in \{1, \dots, s\}$, the quantum 3-simplices of \mathcal{T} . Consider the symmetrization factors $((w_0^j)^{-c_1}((w_1^j)^{c_0})^{N-1/2})$ occuring in the state sums $\mathcal{H}_N^e(T, \tilde{b}, c)$ (see (21), (25), (26) and (28)). Note that by (12) & (13), for an arbitrary quantum 3-simplex (Δ, b, w, f, c) we have

$$\begin{aligned} ((w_0')^{-c_1}(w_1')^{c_0})^{\frac{N-1}{2}} &= \exp\left(\frac{N-1}{2}(-c_1 l_{0,N,*b,c} + c_0 l_{1,N,*b,c})\right) \\ &= \exp\left(\frac{N-1}{2}(-c_1(\frac{l_0}{N} + i\pi f_0) + c_0(\frac{l_1}{N} + i\pi f_1))\right). \end{aligned}$$

We are going to prove

$$(47) \quad \sum_{j=1}^s (-c_1^j(\frac{l_0^j}{N} + i\pi f_0^j) + c_0^j(\frac{l_1^j}{N} + i\pi f_1^j)) = \frac{1}{2N} \langle k_c, L([w; f]) \rangle + \frac{i\pi}{2} \langle r(k_c), i^*(\gamma_2(f)) \rangle_2 \pmod{i\pi}$$

which implies the desired result. To this aim we need several tools of Neumann-Zagier theory, that we recall at first (see [30, 31]). Denote :

- by $C_1(\Delta_j)$ the \mathbb{Z} -module freely generated by the edges e_k^j and $(e'_k)^j$, $k = 0, 1, 2$, of the tetrahedron Δ_j ;
- by \bar{J}_j the quotient module of $C_1(\Delta_j)$ by the relations $e_k^j = (e'_k)^j$, $k = 0, 1, 2$;
- by J_j the quotient module of \bar{J}_j by the relation $e_0^j + e_1^j + e_2^j = 0$.

Let C_1 be the \mathbb{Z} -module freely generated by the edges of T , and put $E(T) := \oplus_j C_1(\Delta_j)$ (the \mathbb{Z} -module of “abstract” edges of T), $\bar{J} := \oplus_j \bar{J}_j$, and $J := \oplus_j J_j$. Consider on these spaces the following 2-forms: the standard inner product $(\ , \)$ on C_1 , $E(T)$ and \bar{J} (each defined with respect to the natural basis given by (cosets of) edges), and the *signed* antisymmetric bilinear form $\langle \ , \ \rangle$ on J given by

$$(48) \quad \langle \ , \ \rangle = \oplus_j *_j \langle \ , \ \rangle_j$$

where $\langle \ , \ \rangle_j$ is the standard antisymmetric bilinear form on J_j , defined in the basis $\{e_0^j, e_1^j\}_j$ by

$$(49) \quad \langle e_0^j, e_1^j \rangle_j = 1 \quad , \quad \langle e_0^j, e_1^m \rangle = 0 \text{ for } m \neq j .$$

Hence $\{e_0^j, e_1^j\}$ is a symplectic basis of $\langle \ , \ \rangle$ restricted to J_j when Δ_j has positive branching orientation $*_j = +1$, and is an anti-symplectic basis of J_j otherwise. Equivalently, $\{e_0^j, e_1^j\}$ is symplectic if the edges e_0^j , $(e'_1)^j$, e_2^j have positive cyclic ordering as viewed from their common endpoint v_0 in Δ_j (in [30] branchings are not needed and not used; one fixes positively cyclically ordered edges e_0^j , e_1^j , e_2^j independently for all Δ_j , so that the signs $*_j$ occuring here are systematically replaced by $+1$). Define linear maps $\beta : C_1 \rightarrow E(T)$ and $\beta^* : E(T) \rightarrow C_1$ on generators by

$$\beta(e) = \sum_{E \rightarrow e} E$$

and

$$\beta^*(e_k^j) = *_j \left(p(e_{k+1}^j) - p(e_{k+2}^j) + p((e'_{k+1})^j) - p((e'_{k+2})^j) \right)$$

and the same for $\beta^*((e'_k)^j)$, where indices are regarded mod(3), and $p : E(T) \rightarrow C_1$ is the identification map, assigning to an abstract edge its coset in T . They induce maps (we keep the same notations)

$$\beta : C_1 \rightarrow J \quad , \quad \beta^* : J \rightarrow C_1 .$$

Clearly, for any edge e of T we have

$$(50) \quad (\beta^*(e_k^j), e) = \langle e_k^j, \beta(e) \rangle .$$

Also, β^* splits as

$$\beta^* : J \xrightarrow{\beta_1} \bar{J} \xrightarrow{\beta_2} C_1$$

where $\beta_1(e_k^j) := *_j(e_{k+1}^j - e_{k+2}^j)$ and $\beta_2(e_k^j) := p(e_k^j) + p((e'_k)^j)$. Similarly to (50), β_1 and the natural projection $q : \bar{J} \rightarrow J$ are adjoint maps: for all $x \in \bar{J}$ and $a \in J$ we have

$$(51) \quad (\beta_1(a), x) = \langle a, q(x) \rangle .$$

It is not hard to see that $\text{Im}(\beta) \subset \text{Ker}(\beta^*)$. Put

$$H := \frac{\text{Ker}(\beta^*)}{\text{Im}(\beta)} .$$

The 2-form $\langle \cdot, \cdot \rangle$ on J descends to H . Since $\text{Ker}(\beta^*) = \text{Im}(\beta)^\perp$, $\langle \cdot, \cdot \rangle$ is non degenerate on $H/\text{Tors}(H)$. In fact, it is shown in [30] that there is an exact sequence

$$0 \longrightarrow H \xrightarrow{(\gamma', \gamma'_2)} H^1(\partial V; \mathbb{Z}) \oplus H^1(V; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{r-i^*} H^1(\partial V; \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 .$$

Hence $\text{Tors}(H)$ is 2-torsion, and taking coefficients in $\mathbb{C}/2\pi i$, the exact sequence shows that γ' yields an isomorphism

$$\gamma' : H \otimes (\mathbb{C}/2\pi i) \longrightarrow H^1(\partial V; \mathbb{C}/2\pi i) .$$

Moreover, denoting by \cdot the intersection product on $H_1(\partial V; \mathbb{Z})$, and by $\gamma = PD \circ \gamma'$ the map γ' followed by Poincaré duality, for all $a, b \in H$ we have

$$(52) \quad \gamma(a) \cdot \gamma(b) = 2\langle a, b \rangle .$$

The map γ is defined as follows. Recall the notations introduced before Definition 4.10. For any 2-face F of ∂T_0 which is a boundary section of the truncated tetrahedron of T_0 corresponding to Δ_j , let us write $F \rightarrow \Delta_j$, and denote by $s_k^{j,F}$ the edge of F which is opposite to the vertex of F that belongs to the edge e_k^j or $(e'_k)^j$, with the positive orientation as viewed from that vertex. Then, the linear map

$$(53) \quad \begin{array}{ccc} \bar{\gamma} : \bar{J} & \longrightarrow & C_1(\partial T_0) \\ e_k^j & \longmapsto & \sum_{F \rightarrow \Delta_j} s_k^{j,F} . \end{array}$$

descends to

$$(54) \quad \gamma : H \rightarrow H_1(\partial V; \mathbb{Z}) .$$

Conversely, let $q : \bar{J} \rightarrow J$ be the natural projection. Denote by $\partial T'_0$ the cellulation of ∂V dual to ∂T_0 . Represent classes in $H_1(\partial V; \mathbb{Z})$ by simplicial loops in $\partial T'_0$. For all $F \rightarrow \Delta_j$, denote by $a_k^{j,F}$ the simplicial arc in $\partial T'_0 \cap F$ which faces the vertex of F that belongs to the edge e_k^j or $(e'_k)^j$, with the positive orientation as viewed from that vertex. Define a linear map

$$\begin{array}{ccc} \bar{\delta} : C_1(\partial T'_0) & \longrightarrow & \bar{J} \\ a_k^{j,F} & \longmapsto & e_k^j . \end{array}$$

Then, $q\bar{\delta}$ descends to a map $\delta : H_1(\partial V; \mathbb{Z}) \rightarrow H$. Moreover, for all $x \in H_1(\partial V; \mathbb{Z})$ we have

$$(55) \quad \gamma \circ \delta(x) = 2x .$$

The identity (52) follows from (55), together with the simple fact that for any simplicial loop C in $\partial T'_0$, we have

$$(56) \quad C \cdot \bar{\gamma}(e_k^j) = *_j \langle \bar{\delta}(C), e_k^j \rangle_j ,$$

where the sign $*_j$ comes from the orientation induced by the basis $\{e_0^j, e_1^j\}$ of J_j , as discussed after (49). Hence γ and δ are adjoint maps with respect to $\langle \cdot, \cdot \rangle$ on J and the intersection form $H_1(\partial V; \mathbb{Z})$: for all $x \in H_1(\partial V; \mathbb{Z})$ and $a \in H$ we have

$$x \cdot \gamma(a) = \langle \delta(x), a \rangle .$$

Taking coefficients in $\mathbb{Z}/2\mathbb{Z}$ and simplicial paths in T'_0 representing normal arcs with respect to T_0 , one defines similarly to δ a map $\delta_2 : H_1(V; \mathbb{Z}/2\mathbb{Z}) \rightarrow H \otimes (\mathbb{Z}/2\mathbb{Z})$. The map γ'_2 in the above exact sequence is defined by

$$\gamma'_2(a)(c) = \langle a, \delta_2(c) \rangle$$

for all $a \in H \otimes (\mathbb{Z}/2\mathbb{Z})$ and $c \in H_1(V; \mathbb{Z}/2\mathbb{Z})$. The maps γ' and γ'_2 produce respectively the boundary and bulk weights of V . For instance, working with coefficients in \mathbb{Z} to simplify notations, it is clear that the \mathbb{Z} -valued boundary weights are given by the map

$$\begin{aligned} \bar{\gamma}'' : \quad \bar{J} &\longrightarrow C^1(\partial T'_0) \\ x &\longmapsto (C \mapsto (x, \bar{\delta}(C))) . \end{aligned}$$

On another hand, denoting by $A \in J$ a representative of $a \in H$, $\bar{\gamma}''$ induces a map

$$\begin{aligned} \gamma'' : \quad H &\longrightarrow H^1(\partial V; \mathbb{Z}) \\ a &\longmapsto \bar{\gamma}''(\beta_1(A)) . \end{aligned}$$

Then, using successively (51), (52) and (55), one can check that $\gamma'' = \gamma'$ as follows:

$$(57) \quad \gamma''(a)([C]) = (\beta_1(A), \bar{\delta}(C)) = \langle a, q\bar{\delta}([C]) \rangle = \frac{1}{2}\gamma(a) \cdot \gamma(\delta([C])) = \gamma(a) \cdot [C] .$$

Consider now the QH triangulation \mathcal{T} . By using the classical log-branch $l = (l_0^j, l_1^j, l_2^j)_j$ and the flattening $f = (f_0^j, f_1^j, f_2^j)_j$, define a vector

$$(58) \quad v_{l,f,N} := \sum_{j=1}^s \left(\frac{l_1^j}{N} + i\pi f_1^j \right) e_0^j - \left(\frac{l_0^j}{N} + i\pi f_0^j \right) e_1^j \in J \otimes \mathbb{C} .$$

Similarly, put

$$(59) \quad v_l := \frac{1}{N} \sum_{j=1}^s l_1^j e_0^j - l_0^j e_1^j \in J \otimes \mathbb{C} .$$

By the log-branch condition on each quantum 3-simplex $(\Delta_j, b^j, w^j, f^j, c^j)$, we have

$$\begin{aligned} \beta_1(v_{l,f,N}) &= \sum_{j=1}^s *_j \left(\left(\frac{l_0^j}{N} + i\pi f_0^j \right) e_0^j + \left(\frac{l_1^j}{N} + i\pi f_1^j \right) e_1^j - \left(\frac{l_0^j + l_1^j}{N} + i\pi(f_0^j + f_1^j) \right) e_2^j \right) \\ &= \sum_{j=1}^s *_j \left(\left(\frac{l_0^j}{N} + i\pi f_0^j \right) e_0^j + \left(\frac{l_1^j}{N} + i\pi f_1^j \right) e_1^j + \left(\frac{l_2^j}{N} + i\pi f_2^j \pm i\pi \right) e_2^j \right) \\ &= \sum_{j=1}^s *_j \left(\left(\frac{l_0^j}{N} + i\pi f_0^j \right) e_0^j + \left(\frac{l_1^j}{N} + i\pi f_1^j \right) e_1^j + \left(\frac{l_2^j}{N} + i\pi f_2^j \right) e_2^j \right) \in J \otimes (\mathbb{C}/2\pi i) . \end{aligned}$$

So $\beta_1(v_{l,f,N})$ represents the map assigning the value $*_j(l_k^j/N + i\pi f_k^j) \bmod(2\pi i)$ to the abstract edges $e_k^j, (e'_k)^j$ of T . The log-branch condition at each edge of T now gives

$$\beta^*(v_{l,f,N}) = \beta_2(\beta_1(v_{l,f,N})) = 0 \bmod(2\pi i) .$$

Hence we can consider the coset (keeping the same notation)

$$v_{l,f,N} \in H \otimes (\mathbb{C}/2\pi i) .$$

By the definition of the boundary c -weight k_c as a cochain on normal loops, it is immediate that for any one of the four 2-faces F of ∂T_0 such that $F \rightarrow \Delta_j$, we have

$$-c_1^j \left(\frac{l_0^j}{N} + i\pi f_0^j \right) + c_0^j \left(\frac{l_1^j}{N} + i\pi f_1^j \right) = k_c \left(\left(\frac{l_1^j}{N} + i\pi f_1^j \right) a_0^{j,F} - \left(\frac{l_0^j}{N} + i\pi f_0^j \right) a_1^{j,F} \right) .$$

Now

$$q\bar{\delta} \left(\sum_{j=1}^s \sum_{F \rightarrow \Delta_j} \left(\frac{l_1^j}{N} + i\pi f_1^j \right) a_0^{j,F} - \left(\frac{l_0^j}{N} + i\pi f_0^j \right) a_1^{j,F} \right) = 4 v_{l,f,N} .$$

By (55) we deduce

$$\sum_{j=1}^s \sum_{F \rightarrow \Delta_j} \left(\left(\frac{l_1^j}{N} + i\pi f_1^j \right) a_0^{j,F} - \left(\frac{l_0^j}{N} + i\pi f_0^j \right) a_1^{j,F} \right) = 2 \gamma(v_{l,f,N}) \bmod(4\pi i) .$$

Hence

$$\begin{aligned} \sum_{j=1}^s 4(-c_1^j(\frac{l_0^j}{N} + i\pi f_0^j) + c_0^j(\frac{l_1^j}{N} + i\pi f_1^j)) &= \sum_{j=1}^s k_c \left(\sum_{F \rightarrow \Delta_j} (\frac{l_1^j}{N} + i\pi f_1^j) a_0^{j,F} - (\frac{l_0^j}{N} + i\pi f_0^j) a_1^{j,F} \right) \\ &= 2 k_c (\gamma(v_{l,f,N})) \bmod(4\pi i) . \end{aligned}$$

The result follows from this, $\gamma(v_{l,f,N}) = \gamma(v_l)/N + i\pi\gamma_2(f)$, and the computation (57), which gives the weights as the Poincaré duals of these classes. \square

Remark 4.18. The symmetrization factor $\alpha_N(\mathcal{T}) = \prod_{j=1}^s (((w_0^j)')^{-c_1} ((w_1^j)')^{c_0})^{N-1/2}$ of the state sum $\mathcal{H}_N^e(\mathcal{T})$ contains the “independent” contribution of the global charge c . In the reduced state sum $(\mathcal{H}_N^e)^R(\mathcal{T})$ introduced in Theorem 1.1 (4), the charge and the flattening cooperate to build the variables $(w_k^j)'$, which verify the tetrahedral equations (19) and the edge equations (33). These are the ultimate arguments of $(\mathcal{H}_N^e)^R(\mathcal{T})$, as well as the class $\exp \circ L_{N,c}([w; f]) \in H^1(\partial V; \mathbb{C}^*)$ is the ultimate boundary weight of the reduced invariant $(\mathcal{H}_N^e)^R(V, \rho, (h, k)) \equiv_{4N} (\mathcal{H}_N^e)^R(\mathcal{T})$. Note that by using the tetrahedral and edge conditions, one realizes at first that the $(w_k^j)'$ carry a class in $H^1(\partial V; \mathbb{C}^*/\{\pm 1\})$; then one realizes that the compatibility between the boundary and bulk weights fixes the sign ambiguity.

Remark 4.19. The Epstein-Penner family of rich components \mathcal{F}_{EP} is only one of the possible natural choices. For instance, for every cusped manifold M and every $m \geq 2$ we can consider the family \mathcal{F}_m of rich components of gluing varieties of weakly branched triangulations of \hat{V} formed by no more than m tetrahedra, or the family \mathcal{F}_{m_0} where m_0 is the minimum m such that $\mathcal{F}_m \neq \emptyset$. For example if M is the complement of the knot 5_2 in S^3 , then $m_0 = 3$, while the Epstein-Penner subdivision is a triangulation made by 4 hyperbolic ideal tetrahedra. In practice, any rich component is suited to the asymptotic analysis of [5], and is pertinent when we are interested to study small deformations of the complete solution.

4.4. Relation with Chern-Simons theory. Recall that we denote by V the compact 3-manifold obtained by completing the cusp of M with a torus. Chern-Simons theory with gauge group $PSL(2, \mathbb{C})$ associates to ∂V the *Chern-Simons bundle* $\mathcal{L}_{\partial V} \rightarrow X(\partial V)$, which is a \mathbb{C}^* -bundle with a canonical connection 1-form and a canonical inner product, and to V the *Chern-Simons section* s_V , which is a parallel section of the pull-back bundle $i^* \mathcal{L}_{\partial V}$ over $X(V)$, where $i : X(V) \rightarrow X(\partial V)$ is the restriction map (the bundle $i^* \mathcal{L}_{\partial V}$ is flat and trivial; see [19, 25] and Remark 4.22 (1) below).

We claim that in the case when the bulk f -weight $h_f = 0$, the bottom row of the diagram of Theorem 4.17 encodes the restriction of the pair $(i^* \mathcal{L}_{\partial V}, s_V)$ to the Zariski open subset $\Omega_{\mathcal{F}_{\text{EP}}} \subset X_0(M)$, identified as a component of $X(V)$. For arbitrary h_f one gets a twisted version of $(i^* \mathcal{L}_{\partial V}, s_V)$. Our proof of this fact relies on a certain number of results that we are going to recall at first. Denote by $\text{CS}(M') \in \mathbb{R}/\mathbb{Z}$ the Chern-Simons invariant of the Levi-Civita connection of a closed riemannian 3-manifold M' , and by $\text{CS}(M)$ its extension to cusped hyperbolic manifolds M , according to Meyerhoff. By definition, $\text{CS}(M)$ is the limit of $\text{CS}(M'_n)$ for any sequence (M'_n) of closed hyperbolic Dehn fillings of M converging to M in Dehn surgery space. In [31], Corollary 14.6 & Theorem 14.7, Neumann proved (using the notations of the present paper):

Theorem 4.20. *Let Y be M or a closed hyperbolic Dehn filling M' of M . Denote by ρ_Y the hyperbolic holonomy of Y , identified as a character of $V \subset M'$ in the case $Y = M'$. If $Y = M$, put $k_Y = 0$, and if $Y = M'$, let k_Y be any boundary f -weight of V relative to ρ_Y such that k_Y vanishes on the meridian of the added solid torus. Then*

$$\mathcal{H}_0(V, \rho_Y, (0, k_Y)) = \exp \left(\frac{2}{\pi} \text{Vol}(Y) + 2\pi i \text{CS}(Y) \right) .$$

On the right hand side the invariants are related to the Chern-Simons section s_V as follows. To any compact closed oriented 3-manifold Y , $PSL(2, \mathbb{C})$ -Chern-Simons theory associates a function $s_Y : X'(Y) \rightarrow \mathbb{C}^*$ defined on the variety $X'(Y)$ of $PSL(2, \mathbb{C})$ -characters of Y . When Y is hyperbolic

with holonomy ρ_Y , a classical result of Yoshida [36] gives

$$(60) \quad s_Y(\rho_Y) = \exp\left(\frac{2}{\pi} \text{Vol}(Y) + 2\pi i \text{CS}(Y)\right).$$

If moreover $Y = V \cup_{\varphi} (D^2 \times S^1)$ is a closed hyperbolic Dehn filling of M whose holonomy ρ_Y , when considered as an augmented character of V that factors to ρ_Y under the Dehn filling, lies in a sufficiently small neighborhood D of ρ_{hyp} in $X(V) = X(M)$, then Kirk-Klassen ([25], pages 554–556) showed that

$$(61) \quad s_Y(\rho_Y) = \langle s_V(\rho_Y), s_{D^2 \times S^1}(\rho_Y) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product of $i^* \mathcal{L}_{\partial V}$, and on the right hand side ρ_Y denotes also the induced augmented character of the glued solid torus $D^2 \times S^1$. Explicitly, if we fix the gauge on $i^* \mathcal{L}_{\partial V}$ by taking as coordinates on D the *standard* logarithms $\log(\lambda)$ and $\log(\mu)$, which are equal to 0 at ρ_{hyp} , we have

$$(62) \quad s_V(\rho_{hyp}) = \exp\left(\frac{2}{\pi} \text{Vol}(M) + 2\pi i \text{CS}(M)\right).$$

Then, if the Dehn filling instruction φ maps the meridian $\partial(D^2 \times *)$ to $l^q m^p$ and the longitude $* \times S^1$ to $l^s m^r$ (so $ps - qr = 1$), the formula (61) splits as

$$(63) \quad s_Y(\rho_Y) = s_V(\rho_{hyp}) \times \exp\left(-\frac{1}{2\pi i} \int_{\rho_{hyp}}^{\rho_Y} (\log(\lambda) d\log(\mu) - \log(\mu) d\log(\lambda))\right) \exp(-(s \log(\lambda) + r \log(\mu))).$$

The first exponential term is the variation of s_V between ρ_{hyp} and ρ_Y (that is, s_V being a parallel section of $i^* \mathcal{L}_{\partial V}$, the holonomy of the connection 1-form of $i^* \mathcal{L}_{\partial V}$ between these points); its product with $s_V(\rho_{hyp})$ is thus the value of $s_V(\rho_Y)$ in the chosen gauge. The second exponential term is the value of $s_{D^2 \times S^1}(\rho_Y)$ in the gauge fixed by the logarithms $\log(\text{hol}_{\partial(D^2 \times *)}) = q \log(\lambda) + p \log(\mu) = 2\pi i$, $\log(\text{hol}_{* \times S^1}) = s \log(\lambda) + r \log(\mu)$ corresponding to the Dehn filling. Its argument is $-(\text{length} + i \text{rotation angle})$ of the geodesic core of the surgery torus. The arguments of both exponential terms can be collected in a single integral by changing the coordinates $\log(\lambda)$, $\log(\mu)$ on D to $\log(\lambda) + 2\pi i r$, $\log(\mu) - 2\pi i s$, that is, by working on a different leaf of the Riemann surface of these maps. Then the gauge $\log(\text{hol}_{\partial(D^2 \times *)}) = 0$, like the vanishing value of k_Y in Theorem 4.20.

We are going to show that $(i^* \mathcal{L}_{\partial V}, s_V)|_{\Omega_{\mathcal{F}_{\text{EP}}}}$ is determined by the function $\mathcal{H}_0^{\mathcal{F}_{\text{EP}},0} : A_{0,\mathcal{F}_{\text{EP}}}(M)_{\infty} \rightarrow \mathbb{C}$. Because of Theorem 4.20 and (62), it is enough to identify the variation of $\mathcal{H}_0^{\mathcal{F}_{\text{EP}},0}$ with the one of s_V , given in (63). This is the content of the following result.

Proposition 4.21. *The function $\mathcal{H}_0^{\mathcal{F}_{\text{EP}},0}$ descends to a parallel section $\mathcal{S}^{\mathcal{F}_{\text{EP}},0}$ of a (flat and trivial) \mathbb{C}^* -bundle $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$ with a canonical connection 1-form and a canonical inner product, such that $\mathfrak{h}^*(\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M), \mathcal{S}^{\mathcal{F}_{\text{EP}},0})$ is isomorphic to $(i^* \mathcal{L}_{\partial V}, s_V)|_{\Omega_{\mathcal{F}_{\text{EP}}}}$.*

Proof. We derive the bundle $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$ from the variation of the invariant of patterns $\mathcal{H}_0^{\mathcal{F}_{\text{EP}}}$, like the Chern-Simons line bundle is derived from the Chern-Simons action in [19, 25]. Hence we define $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$ as the quotient space of $A_{0,\mathcal{F}_{\text{EP}}}(M)_{\infty} \times \mathbb{C}^*$ under the action of \mathbb{Z}^2 given by

$$(64) \quad \forall a, b \in \mathbb{Z}, (a, b) \cdot ([\lambda; p], [\mu; q], z) := ([\lambda; p + a], [\mu; q + b], ze^{b \log(\lambda) - a \log(\mu)}).$$

We have a \mathbb{C}^* -bundle projection

$$\pi'_{\infty} : \begin{array}{ccc} \mathcal{L}_{\mathcal{F}_{\text{EP}}}(M) & \longrightarrow & A_{0,\mathcal{F}_{\text{EP}}}(M) \\ [[\lambda; p], [\mu; q], z] & \longmapsto & (\lambda, \mu) \end{array}$$

Denote by $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)^{-1}$ the inverse \mathbb{C}^* -bundle; by (64) it is the same as $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(-M)$, where $-M$ denotes M with reversed orientation. By considering \mathbb{C}^* as the bundle over a point, we have a well-defined bundle map

$$\langle \cdot, \cdot \rangle : \begin{array}{ccc} \mathcal{L}_{\mathcal{F}_{\text{EP}}}(M) \times \mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)^{-1} & \longrightarrow & \mathbb{C}^* \\ (([\lambda; p], [\mu; q], z_1), ([\lambda; p], [\mu; q], z_2)) & \longmapsto & z_1 z_2 \end{array}$$

It defines an inner product on $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$. Also, the restriction to $A_{0,\mathcal{F}_{\text{EP}}}(M)_{\infty}$ of the 1-form on $\mathbb{C} \times \mathbb{C}$ given by (we put a coordinate $l_a(\lambda) := \log(\lambda) + 2\pi i a$ on each copy of \mathbb{C} , viewed as the Riemann surface of \log)

$$(65) \quad \eta := -\frac{1}{2\pi i} (l_p(\lambda)dl_q(\mu) - l_q(\mu)dl_p(\lambda))$$

defines a flat analytic connection 1-form on $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$.

By Lemma 4.15, given any even class $\theta \in H^1(\partial V; 2\mathbb{Z})$ and any point $[w; f]_Z \in \Omega_{\mathcal{F}_{\text{EP}}}(Z)_{\infty,0}$, the flattening f can be modified to a flattening f' such that $[w; f']_Z \in \Omega_{\mathcal{F}_{\text{EP}}}(Z)_{\infty,0}$ and the corresponding boundary weights satisfy

$$L([w; f']_Z) - L([w; f]_Z) = \pi i \theta .$$

As in the proof of Theorem 1.1, denote by v_l (resp. $v_{l'}$) the vector in $H \otimes \mathbb{C}$ associated to the classical log-branch at $[w; f]_Z$ (resp. $[w; f']_Z$), and similarly denote by $v_d \in H \otimes (\mathbb{C}/2\pi i)$ the vector associated to $(\log(w_0^j), \log(w_1^j), \log(w_2^j))_j$. Recall the class $d_w \in H^1(\partial V; \mathbb{C}/2\pi i)$ in (36). We have

$$d_w = \gamma'(v_d)$$

and

$$\pi i \theta = \gamma'(v_{l'} - v_l) = \gamma' \left(\pi i \sum_{j=1}^s ((f')_1^j - f_1^j) e_0^j - ((f')_0^j - f_0^j) e_1^j \right) .$$

Hence

$$(66) \quad \begin{aligned} \mathcal{H}_0^{\mathcal{F}_{\text{EP}}}([w; f']_Z) \mathcal{H}_0^{\mathcal{F}_{\text{EP}}}([w; f]_Z)^{-1} &= \exp \left(\sum_{j=1}^s *_{j_1} \left(((f')_1^j - f_1^j) \log(w_0^j) - ((f')_0^j - f_0^j) \log(w_1^j) \right) \right) \\ &= \exp \left(-\frac{1}{\pi i} \langle v_{l'} - v_l, v_d \rangle \right) \\ &= \exp \left(-\frac{1}{2\pi i} \langle \gamma'(v_{l'} - v_l), \gamma'(v_d) \rangle \right) \\ &= \exp \left(\frac{1}{2} (\theta(m) \log(\text{hol}_l(\rho(w))) - \theta(l) \log(\text{hol}_m(\rho(w)))) \right) \end{aligned}$$

where the map hol_{γ} , γ a curve in ∂V , is defined in (31). The first equality follows from (22) and (27), the second is by the definition (48) of the symplectic pairing on $H \otimes \mathbb{C}$, the third follows from (52) and Poincaré duality, and the last one uses the fact that the symplectic basis of $H_1(\partial V; \mathbb{Z})$ is given by l and m (in the order). By setting $\theta(m) := 2b$ and $\theta(l) := 2a$, we see from (64) and (66) that the function $\mathcal{H}_0^{\mathcal{F}_{\text{EP}}, h_f}$ (with arbitrary bulk weight h_f for the moment) descends from $A_{0,\mathcal{F}_{\text{EP}}}(M)_{\infty}$ to a section of $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$, that we denote by

$$\mathcal{S}^{\mathcal{F}_{\text{EP}}, h_f} : A_{0,\mathcal{F}_{\text{EP}}}(M) \rightarrow \mathcal{L}_{\mathcal{F}_{\text{EP}}}(M) .$$

Next we show that this section is parallel with respect to the connection η . The method is similar to the computation leading to (66). As remarked in [30], Lemma 10.2, for any \mathbb{Q} -vector space E , the skew symmetric bilinear form $\langle \cdot, \cdot \rangle$ on J induces a symmetric bilinear map

$$\begin{aligned} B : \quad (H \otimes E) \otimes (H \otimes E) &\longrightarrow E \wedge E \\ (a \otimes v) \otimes (b \otimes w) &\longmapsto \langle a, b \rangle v \wedge w . \end{aligned}$$

Consider the map \cdot_E on $H_1(\partial V; E) = H_1(\partial V; \mathbb{Z}) \otimes E$ induced by the intersection product on $H_1(\partial V; \mathbb{Z})$:

$$\begin{aligned} \cdot_E : \quad H_1(\partial V; E) \otimes H_1(\partial V; E) &\longrightarrow E \wedge E \\ (x \otimes v) \otimes (y \otimes w) &\longmapsto (x \cdot y) v \wedge w . \end{aligned}$$

Denote again by γ the map $\gamma \otimes \text{id} : H \otimes E \rightarrow H_1(\partial V; E)$. By (52), we have a commutative diagram

$$\begin{array}{ccc} (H \otimes E) \otimes (H \otimes E) & \xrightarrow{B} & E \wedge E \\ \gamma \otimes \gamma \downarrow & & \downarrow 2\times \\ H_1(\partial V; E) \otimes H_1(\partial V; E) & \xrightarrow{\cdot E} & E \wedge E \end{array}$$

In particular, let E be the space of functions on the covering \mathcal{W}_∞ of \mathbb{C}_* (see (14)). Denote by $l_0^j, l_1^j \in E$ the classical log-branch functions at the edges e_0^j, e_1^j of Δ_j . Put $a = \sum_{j=1}^s e_0^j \otimes l_1^j - e_1^j \otimes l_0^j \in H \otimes E$. For any point $[w; f]_Z \in \Omega_{\mathcal{F}_{\text{EP}}}(Z)_{\infty,0}$ we have $\gamma(a)([w; f]_Z) = l \otimes L([w; f]_Z)(m) - m \otimes L([w; f]_Z)(l)$; dually the class $\gamma'(a) \in H^1(\partial V; E)$, $\gamma'(a) = PD \circ \gamma(a)$, is given by $\gamma'(a)([w; f]_Z) = L([w; f]_Z) \in H^1(\partial V; \mathbb{C})$. Moreover, in $E \wedge E$ we have

$$2 \sum_{j=1}^s *_{\gamma} (l_0^j \wedge l_1^j) = B(a, a) = \frac{1}{2} \gamma(a) \cdot_E \gamma(a) = L(\cdot)(l) \wedge L(\cdot)(m) .$$

By applying the homomorphism $E \wedge E \rightarrow \Omega^1(\mathcal{W}_\infty)$, $f \wedge g \mapsto f dg - g df$, we deduce

$$2 \sum_{j=1}^s *_{\gamma} (l_0^j dl_1^j - l_1^j dl_0^j) = L(\cdot)(l) dL(\cdot)(m) - L(\cdot)(l) dL(\cdot)(m) .$$

Hence, for any two points $[w; f]_Z, [w'; f']_Z$ connected by a continuous path $(\gamma^1, \dots, \gamma^s)$ in $\Omega_{\mathcal{F}_{\text{EP}}}(Z)_{\infty,0}$, we have

$$(67) \quad \mathcal{H}_0^{\mathcal{F}_{\text{EP}}}([w'; f']_Z) \mathcal{H}_0^{\mathcal{F}_{\text{EP}}}([w; f]_Z)^{-1} = \exp \left(2 \sum_{j=1}^s *_{\gamma} \int_{\gamma^j} \eta \right) = \exp \left(\int_{\gamma} \eta \right)$$

where $\gamma \subset A_{0, \mathcal{F}_{\text{EP}}}(M)_\infty$ is the image of $(\gamma^1, \dots, \gamma^s)$ under the map loghol_∞ . Hence $\mathcal{S}^{\mathcal{F}_{\text{EP}}, h_f}$ is a parallel section of $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$. By a simple rescaling of coordinates (namely, multiplication by $4\pi i$), we can identify $\mathfrak{h}^* \mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$ with the restriction of $i^* \mathcal{L}_{\partial V}$ to $\Omega_{\mathcal{F}_{\text{EP}}}$ (see [25], end of Section 3 and pages 555-556). Finally, as described before the statement, we see from (63) and (67) that $\mathfrak{h}^* \mathcal{S}^{\mathcal{F}_{\text{EP}}, 0} = (s_V)_{|\Omega_{\mathcal{F}_{\text{EP}}}}$. \square

Remark 4.22. 1) In [25], one defines the Chern-Simons bundle over the variety of characters $X'(\partial V)$ instead of the variety of augmented characters $X(\partial V)$; as coordinates on $X'(\partial V)$ one takes $1/2\pi i$ times the logarithms of the eigenvalues at a pair of meridian and longitude curve on ∂V . A fundamental domain for the action of $\mathbb{Z} \times \mathbb{Z}$ on the space \mathbb{C}^2 of coordinates is $([0; 1/2] \times \mathbb{R}) \times ([0; 1] \times \mathbb{R})$. The rescaling factor $4\pi i$ at the end of the proof above replaces these coordinates by ours, the logarithms of the squared eigenvalues, so that a fundamental domain for the action of $\mathbb{Z} \times \mathbb{Z}$ on $A_{0, \mathcal{F}_{\text{EP}}}(M)_\infty$ is $([0; 2i\pi] \times \mathbb{R})^2$.

2) The bundle $\mathcal{L}_{\mathcal{F}_{\text{EP}}}(M)$ is the restriction to $A_{0, \mathcal{F}_{\text{EP}}}(M)$ of a bundle $\mathcal{L}(T^2)$ over $X(T^2) = X(\partial V)$ defined by the very same action (64). Of course the bundle $\mathcal{L}(T^2)$ is isomorphic to $\mathcal{L}_{\partial V}$. Its connection 1-form (65) has curvature $F_\eta = (-1/i\pi) d \log(\lambda) \wedge d \log(\mu)$, its Euler class is represented by $(-1/2(i\pi)^2) d \log(\lambda) \wedge d \log(\mu)$, and its Euler number is -2 .

5. QHFT PARTITION FUNCTIONS

Let us consider for simplicity the patterns \mathcal{P} with topological support $Y_{\mathcal{P}} = (W, L, h_c)$. Hence $V = W$ is a compact closed oriented connected 3-manifold, and only the bulk c -weight h_c occurs.

In order to build the corresponding analytic configurations we specialize the quasi-regular triangulations T of W considered in Proposition 2.3: we require that L can be realized by a *Hamiltonian* subcomplex H of the 1-skeleton of T . Such a pair (T, H) is called a *distinguished* triangulation of (W, L) . By definition, a *global charge c on (T, H)* is a rough global charge on T satisfying the following additional global constraints on the total edge charges:

- For every edge e of T not contained in H , we have $C(e) = 2$;

- For every edge e of T contained in H , we have $C(e) = 0$.

Note that a global charge c on (T, H) encodes H (i.e. the link L). Let us turn to the encoding of the whole topological support $Y_{\mathcal{P}}$. The next two results are similar to Proposition 4.8 and 4.9.

Proposition 5.1. (1) Every pair (W, L) has quasi-regular distinguished triangulations (T, H) .
 (2) Every global charge c on (T, H) determines a bulk c -weight h_c of W .
 (3) For every bulk c -weight h_c of W and every distinguished triangulation (T, H) of (W, L) there is a global charge c on (T, H) with bulk c -weight equal to h_c .

Let us fix a quasi-regular distinguished triangulation (T, H) of (W, L) and a global charge c on it. Fix a branching b on T (it exists always). Consider the associated coarse analytic configurations $\mathcal{A}_N^C(T, b, c)$. Note that, for every odd $N \geq 3$, the defining equations of the algebraic variety $G_0(T, b, c)_N$ are:

- For every edge e of T not contained in H , $W'(e) = \zeta_N^{-1}$;
- For every edge e of T contained in H , $W'(e) = 1$.

We have:

Proposition 5.2. (1) Every point $[w; f] \in G_0(T, b)_\infty$ determines a character $\rho = \rho(w)$ and a bulk f -weight h_f , such that $\mathcal{P} = (W, L, \rho, h_f, h_c)$ is a pattern with topological support $Y_{\mathcal{P}}$.
 (2) For every pattern $\mathcal{P} = (W, L, \rho, h_f, h_c)$ there is a point $[w; f] \in G_0(T, b)_\infty$ with holonomy ρ and bulk f -weight h_f .

Finally the construction of the quantum hyperbolic invariants of the topological support (W, L, h_c) is achieved by the following Proposition. Similarly to Theorem 4.16, it relies essentially on the results of [1, 2], adapted to the setup of Section 3.3.

Proposition 5.3. For every pattern $\mathcal{P} = (W, L, \rho, h_f, h_c)$, every distinguished quasi-regular branched triangulation (T, b, H) of (W, L) , every global charge c on (T, H) with bulk c -weight equal to h_c , every point $[w; f] \in G_0(T, b)_\infty$ with holonomy ρ and bulk f -weight h_f , and every odd $N \geq 3$, the scalars

- $\mathcal{H}_0(\mathcal{P}) := \pm \mathcal{H}_0(T, b)([w; f])$
- $\mathcal{H}_N(\mathcal{P}) := {}_{2N} N^{2-v} \mathcal{H}_N(T, b, c)([w; f])$

are well defined invariants of the pattern \mathcal{P} (where v denotes the number of vertices of T). In other words, the value of these scalars does not depend on the arbitrary choice of analytic configuration $\mathcal{A}_N^C(T, b, c)$ up to the respective phase ambiguity.

By definition, for every odd $N \geq 3$, the analytic configuration $\mathcal{A}_N(Y_{\mathcal{P}})$ of the topological support $Y_{\mathcal{P}} = (W, L, h_c)$ consists of the family of analytic configurations $\mathcal{A}_N^C(T, b, c)$ over the globally charged branched triangulations (T, b, c) as in Proposition 5.3.

Remark 5.4. Other choices of the normalization factor N^{2-v} (for instance N^{-v}) would work as well. Accordingly with [4], the present normalization is the right one with respect to Theorem 1.5.

We can compute the symmetrization factor $\alpha(T, b, c)$ of the above function $\mathcal{H}_N(T, b, c)$ as in point (4) of Theorem 1.1. Indeed, looking at the proof we see that when V belongs to the topological support of a QHFT pattern, and T is a triangulation of the closed pseudo-manifold \hat{V} , one can work in the very same way by replacing V with the 3-manifold \bar{V} obtained by removing an open cone neighborhood of each vertex of T . In case $V = W$, $\partial \bar{V}$ is made of 2-spheres only, so that the boundary weights vanish. Then the statement with V replaced by \bar{V} implies:

Corollary 5.5. We have $\alpha(T, b, c) \equiv_2 1$. Hence $\mathcal{H}_N(W, L, \rho) \equiv_{2N} (\mathcal{H}_N)^R(W, L, \rho)$.

For QHFT patterns \mathcal{P} with topological support more general than (W, L, h_c) , the situation is more elaborated; in particular, one requires that H is Hamiltonian in the 1-skeleton of T from which the non-manifold vertices have been removed, while in the normalization factor N^{2-v} , v denotes the number of manifold (i.e. internal) vertices. Anyway, the results are basically of the same nature. Like for cusped manifolds, we can relax the construction by using weak branchings and the enhanced QH state sums.

6. ON THE INVARIANCE OF ENHANCED QH STATE SUMS

As every QH state sum is ultimately the total contraction of a tensor network supported by a \mathcal{N} -graph, it is convenient to recall at first the main features of a “calculus” based on these diagrams.

6.1. \mathcal{N} -graph calculus. Recall that every \mathcal{N} -graph (Γ, Θ) encodes a weakly branched triangulation (T, \tilde{b}) of some \hat{V} , hence also the induced pre-branched triangulation (T, σ_b) .

6.1.1. \mathcal{N} -graphs representing the same weakly branched triangulation. Taking into account the \mathcal{N} -graph decoding of Section 2, it is easy to see that two \mathcal{N} -graphs (Γ, θ) and (Γ', θ') encode the same triangulation (T, \tilde{b}) if and only if they are equivalent under the equivalence relation generated by:

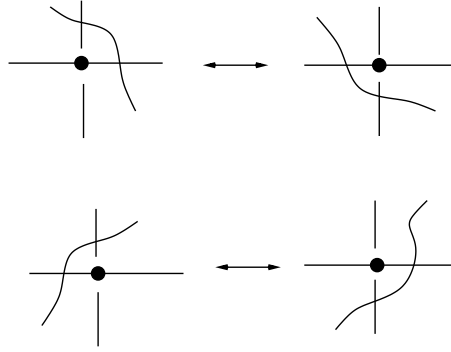


FIGURE 9.

- Plane isotopy.
- Switching the over/under arcs at an accidental crossing.
- The \mathcal{N} -graph versions of oriented Reidemeister moves, called *R-moves* (i.e. formally the usual ones if only accidental crossings are involved, or the Reidemeister move of type III that we show in Figure 9 (the orientation being understood) when one dotted crossing is involved).

6.1.2. Changing the weak branching. Given a \mathcal{N} -graph (Γ, θ) that represents a weakly branched triangulation (T, \tilde{b}) , and an arbitrary change of the weak branching $(T, \tilde{b}) \rightarrow (T, \tilde{b}')$, we want to describe a systematic way to modify (Γ, θ) to a \mathcal{N} -graph (Γ', θ') which represents (T, \tilde{b}') . It will be also useful to specialize such a construction when $\sigma_{\tilde{b}} = \sigma_{\tilde{b}'}$. We begin by treating the local changes of branchings on a given tetrahedron.

The S_4 action $(\Delta, b) \rightarrow (\Delta, b_\beta)$. Consider any branched tetrahedron (Δ, b) with b -ordered vertices v_0, v_1, v_2, v_3 and 2-faces F_0, F_1, F_2, F_3 , as usual. Let $S(J_4)$ be the symmetric group on the set $J_4 = \{0, 1, 2, 3\}$. There is a natural 1–1 correspondence between the elements $\beta \in S(J_4)$ and the branchings b_β on Δ so that $b = b_{\text{Id}}$. The branching b induces a branching $b(i)$ on each face F_i , that is an ordering of its vertices by the indices in $J_3 = \{0, 1, 2\}$. For every $\beta \in S(J_4)$, also the branching b_β induces a branching $b_\beta(i)$ on F_i (note that these faces are still ordered with respect to b , not b_β), and the transition from $b(i)$ to $b_\beta(i)$ is encoded by a permutation $\epsilon_\beta(i) \in S(J_3)$. Clearly we have:

Lemma 6.1. *The branching b_β does not modify the transverse co-orientation at the face F_i given by b if and only if $\epsilon_\beta(i) \in A(J_3)$.*

Let $C(b)$ denote a dotted \mathcal{N} -graph crossing that encodes (Δ, b) as in Figure 3. We consider $C(b)$ as a 2–2 \mathcal{N} -tangle, which can appear in some \mathcal{N} -graph (Γ, θ) representing (T, \tilde{b}) . We want to describe an algorithm that produces for every $\beta \in S(J_4)$ a new decorated 2–2 \mathcal{N} -tangle $C(b_\beta)$ having the following properties:

- $C(b_\beta)$ contains one dotted crossing.
- The four end-points of $C(b_\beta)$ are labelled by elements of $S(J_3)$.

- $C(b_\beta)$ is suited to locally replace the tangle $C(b)$ in every \mathcal{N} -graph (Γ', θ') that represents a change of weak branching $\tilde{b} \rightarrow \tilde{b}'$ on T acting as $b \rightarrow b_\beta$ on the given (Δ, b) (considered as a tetrahedron of T). The $S(J_3)$ labels will eventually contribute to the edge coloring of (Γ', θ') .

Auxiliary construction. Consider the two *basic* $2-2$ tangles made by two oriented simple arcs a, a' that either are disjoint or intersect transversely at one point, without any over/under information. Let us J_4 -label the 4 free end-points of $a \cup a'$ by respecting the following conditions:

- (1) The endpoints labelled by 1, 3 (resp. 0, 2) belong to different arcs and are both either the initial or the final endpoint of the corresponding arc.
- (2) 3, 2 (resp. 1, 0) label the endpoints of the same arc.

Note that these conditions are satisfied by the labellings of the tangle of Figure 3; recall that in this figure, $*_b = 1$ (resp. $*_b = -1$) if and only if 1, 3 label initial (resp. final) endpoints, and that in both cases the arc with endpoint labels 0, 1 passes over the other arc. By the same rule we give every basic tangle with an admissible J_4 -labelling, denoted by \mathcal{B} , a sign $* = \pm 1$. Now we convert \mathcal{B} into a $2-2$ \mathcal{N} -tangle, as follows:

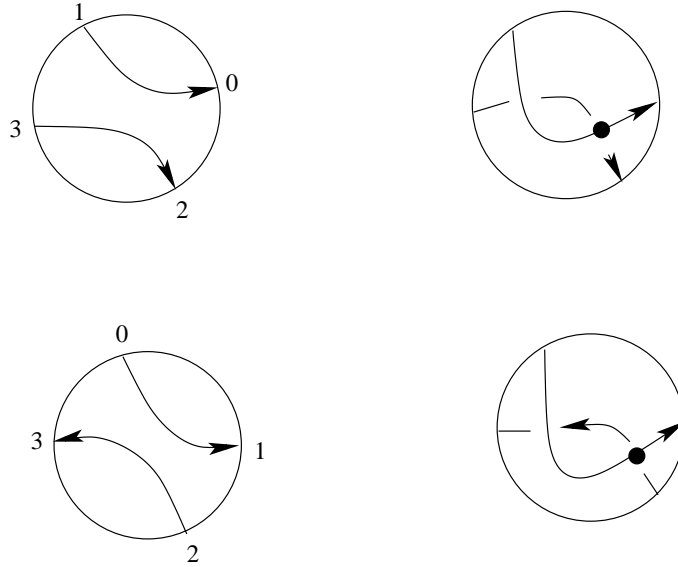
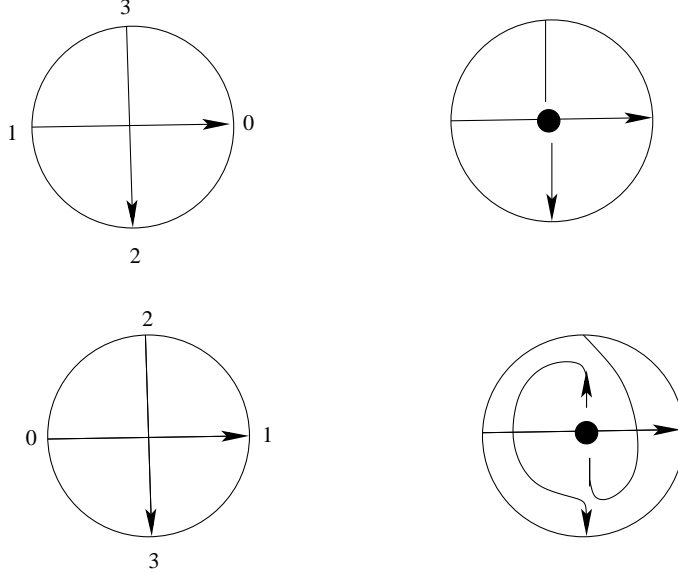


FIGURE 10. Basic to \mathcal{N} -tangles.

- If $a \cap a' = \emptyset$, then perform an oriented Reidemeister move of type II of the arc with labels 0, 1 over the other arc, creating in this way two crossings. Finally there is only one way to put a solid dot on one of these crossings in such a way that the crossing sign and $*$ agree (see some examples in Figure 10).
- If a and a' cross at one point, let A be the arc with 0, 1 labels, and I a small open interval in the interior of the arc with 2, 3 labels, such that I contains the crossing point. Make A passing over I , put a dot at the crossing point, and orient I in such a way that the crossing sign and $*$ agree. Finally we can complete $I \cup A$ to an \mathcal{N} -tangle with the same end-points of $a \cap a'$, by introducing either 0 or 2 accidental crossings on opposite sides of A with respect to $I \cap A$ (see some examples in Figure 11).

The algorithm for the tangle $C(b_\beta)$:

- (1) J_4 -label the free endpoints of the crossing arcs in $C(b)$ as in Figure 3. Forget the over/under information and the arc orientations. Replace the dotted crossing of $C(b)$ with a blank disk D .
- (2) Change the J_4 -labelling of the endpoints according to β .

FIGURE 11. More basic to \mathcal{N} -tangles.

- (3) Note that $*_{b_\beta} = *_b$ if and only if $\beta \in A(J_4)$. Hence we know the sign $*_{b_\beta}$. Let us orient the four edges of the J_4 -labelled tangle constructed in the previous steps in such a way that 1, 3 label initial (resp. final) endpoints if and only if $*_{b_\beta} = 1$ (resp. $*_{b_\beta} = -1$). There is a unique way to complete this picture within the blank disk D , in order to get a decorated basic tangle \mathcal{B} .
- (4) Convert \mathcal{B} into the associated \mathcal{N} -graph tangle, denoted $C'(b_\beta)$, by applying the above auxiliary construction.
- (5) To finish the construction of $C(b_\beta)$, it remains to give each endpoint of $C'(b_\beta)$ a suitable $S(J_3)$ label. Remember the J_4 -labelling of the free endpoints of $C'(b_\beta)$ according to the *initial* branching b . For every label i we will specify the corresponding label $\hat{\epsilon}_\beta(i) \in S(J_3)$. Before Lemma 6.1, we have associated to every arc e_i (dual to a 2-face F_i of (Δ, b)) an element $\epsilon_\beta(i) \in S(J_3)$. Assume that $\epsilon_\beta(i) \in A(J_3)$. Then set $p(i) = 1$ if e_i is ingoing at the crossing of $C(b)$, and $p(i) = -1$ if e_i is outgoing; then define

$$\hat{\epsilon}_\beta(i) := \epsilon_\beta(i)^{p(i)} .$$

If $\epsilon_\beta(i)$ does not belong to $A(J_3)$, then set

$$\hat{\epsilon}_\beta(i) := \epsilon_\beta(i) .$$

The output $C(b_\beta)$ of the algorithm consists of $C'(b_\beta)$ enhanced by the $S(J_3)$ -edge decoration that we have just defined. \square

Every transition $C(b) \rightarrow C(b_\beta)$ is called a *C-move*. In particular, consider the four branched tetrahedra of Figure 2 that have the same pre-branching; start with a \mathcal{N} -tangle associated to the top/left tetrahedron like in Figure 3; apply to it the above algorithm in order to get the \mathcal{N} -tangles for the others. The result (up to plane isotopy and R -moves) is shown in Figure 12. We will call *oriented-C-moves* these local moves at a vertex of an \mathcal{N} -graph.

Globalization. Let (Γ, θ) be an \mathcal{N} -graph encoding (T, \tilde{b}) , \tilde{b}' another weak branching on T . We want to produce an \mathcal{N} -graph (Γ', θ') that encodes (T, \tilde{b}') . For every abstract branched tetrahedron (Δ_j, b_j) of (T, \tilde{b}) there is $\beta_j \in S(J_4)$, such that $\tilde{b} \rightarrow \tilde{b}'$ restricted to Δ_j is given by $b_j \rightarrow (b_j)_{\beta_j}$. By applying at every crossing of (Γ, θ) the above algorithm producing $C'((b_j)_{\beta_j})$, we get a diagram Γ' . Forgetting the orientations, there is a natural 1 – 1 correspondence between the edges of (Γ, θ) and the edges of Γ' . In order to eventually produce (Γ', θ') it remains to fix the edge $\mathbb{Z}/3\mathbb{Z}$ colors. Let e be an edge of

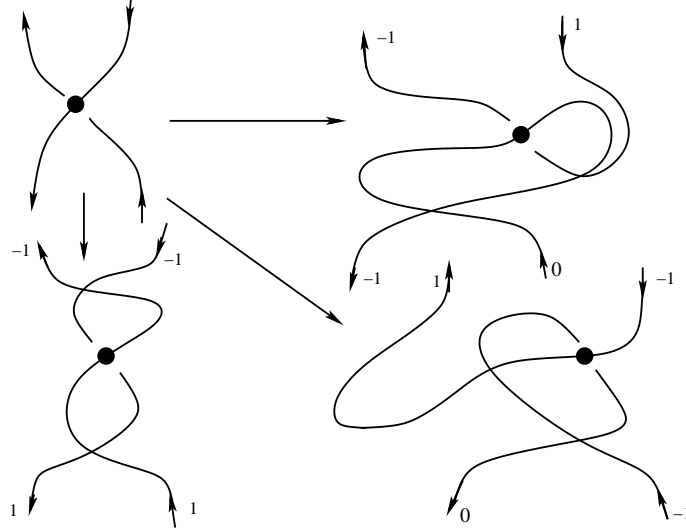


FIGURE 12. Oriented C-moves.

(Γ, θ) equipped with its own color $r(e)$. Denote by \bar{e} the edge of Γ' corresponding to e . This oriented edge \bar{e} is labelled by an “initial” $\hat{e}^i(\bar{e})$ and a “final” $\hat{e}^f(\bar{e})$ permutation in $S(J_3)$, according to the last step of the above algorithm. There is an alternative:

- Both $\hat{e}^i(\bar{e})$ and $\hat{e}^f(\bar{e})$ belong to $A(J_3)$ (in such a case e and \bar{e} have compatible orientations), then the color $r(\bar{e})$ in (Γ', θ') is defined by

$$\hat{e}^f(\bar{e}) \circ (012)^{r(e)} \circ \hat{e}^i(\bar{e}) = (012)^{r(\bar{e})}.$$

- Both $\hat{e}^i(\bar{e})$ and $\hat{e}^f(\bar{e})$ do not belong to $A(J_3)$ (in such a case e and \bar{e} have opposite orientations), then the color $r(\bar{e})$ in (Γ', θ') is defined by

$$\hat{e}^f(\bar{e}) \circ (012)^{-r(e)} \circ \hat{e}^i(\bar{e}) = (012)^{r(\bar{e})}.$$

Note that the oriented C -moves can be freely performed at each crossing, producing new weak branchings that share the same pre-branching. On the other hand, general C -moves that combine to produce a weak branching possibly with a different associated pre-branching have global constraints.

6.1.3. Enhanced MP and bubble moves. It is well-known that two “naked” triangulations T and T' of \hat{V} can be connected by a finite sequence of 3-dimensional Pachner’s moves also called MP or bubble moves respectively. The MP moves are illustrated in Figure 13 in terms of triangulations and dual spines. According to the Figure, $T \rightarrow T'$ ($P \rightarrow P'$) is the *positive* (“ $2 \rightarrow 3$ ”) move. Similarly we have the positive (“ $0 \rightarrow 2$ ”) bubble move and the corresponding negative one.

Let us enhance these moves for structured triangulations.

Definition 6.2. Given any move (either bubble or MP, positive or negative) $T \rightarrow T'$ there is a maximal subset \mathcal{F} of the 2-faces of T that “persist” in T' . Let (T, σ) and (T', σ') be pre-branched triangulations of \hat{V} . Then a move $T \rightarrow T'$ is enhanced to a *pre-branching transit* $(T, \sigma) \rightarrow (T', \sigma')$ if and only if σ and σ' coincide on every 2-face in \mathcal{F} . We consider on the set of pre-branched triangulations of \hat{V} the equivalence relation (denoted by “ \cong_{pb} ”) generated by the pre-branching transits. This equivalence relation lifts to weakly branched triangulations: if (T, \tilde{b}) and (T', \tilde{b}') are weakly branched triangulations, then $T \rightarrow T'$ is enhanced to a \tilde{b} -transit $(T, \tilde{b}) \rightarrow (T', \tilde{b}')$ if it induces a pre-branching transit $(T, \sigma_{\tilde{b}}) \rightarrow (T', \sigma_{\tilde{b}'})$. If (T, b) and (T', b') are branched triangulations, then $T \rightarrow T'$ is enhanced to a b -transit $(T, b) \rightarrow (T', b')$ if b and b' coincide on the edges of \mathcal{F} .

The b -transits have been widely used in [1, 2, 3]. Clearly, a b -transit induces a \tilde{b} -transit. All this can be reformulated in terms of \mathcal{N} -graphs. For example, in Figure 14 we show the different bubble

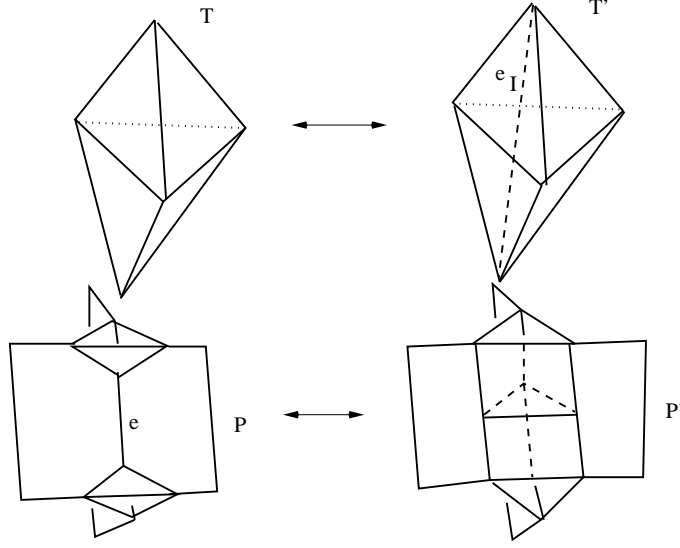
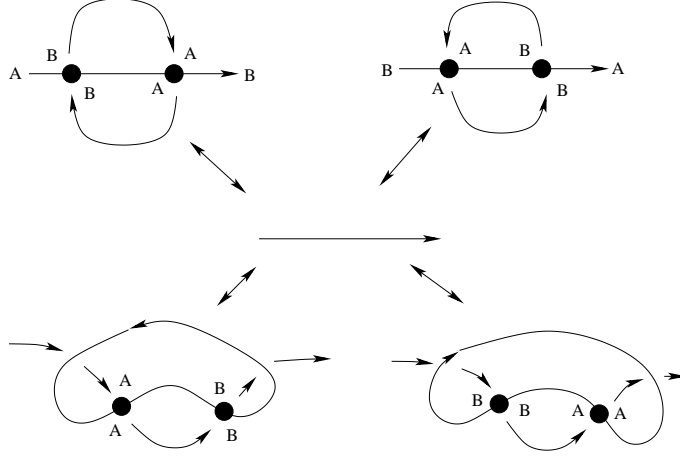


FIGURE 13. The MP move.

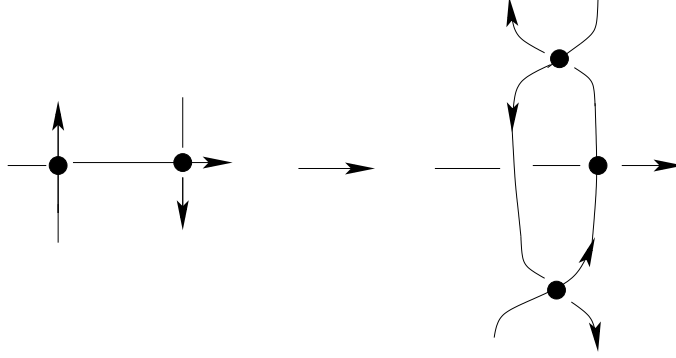
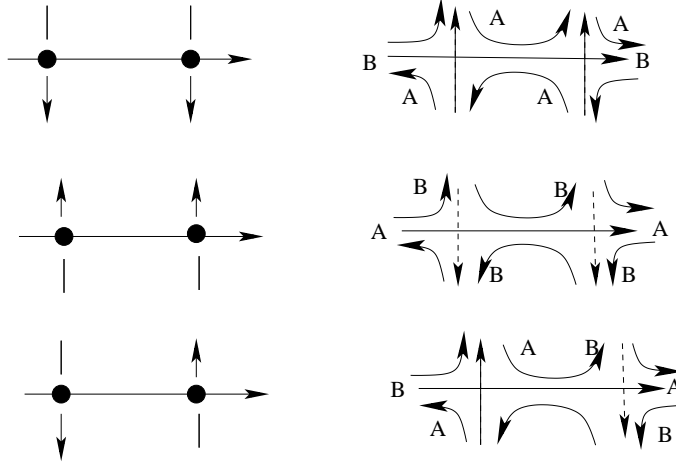
b -transits in terms of normal \mathcal{N} -graphs (recall Remark 2.9) (the labels A , B refer to the type of the corresponding dual square edges according to Figure 7). In Figure 15 we show an example of MP b -transit; note that the 5 dotted crossings involved in the transit have $*_b = 1$. This so-called *Shaeffer's b -transit* plays a distinguished role in the study of matrix dilogarithms (see [2]).

FIGURE 14. Bubble b -transits.

The following Lemma will be useful in applications to the QH state sums in Section 6.2.

Lemma 6.3. *The equivalence relation \cong_{pb} on weakly branched triangulations (resp. \mathcal{N} -graphs) of \hat{V} is generated by the following two transformations:*

- *To change the weak-branching $(T, \tilde{b}) \rightarrow (T, \tilde{b}')$ preserving the pre-branching (resp. to perform oriented C -moves).*
- *To perform \tilde{b} -transits $(T, \tilde{b}) \rightarrow (T', \tilde{b}')$ (either bubble or MP, positive or negative) that locally look like b -transits; “locally” means that we are looking at the sub-patterns of tetrahedra of (T, \tilde{b}) and (T', \tilde{b}') involved in the transit, forgetting the rest of the triangulations (resp. to perform transits of \mathcal{N} -graphs $(\Gamma, \theta) \rightarrow (\Gamma', \theta')$ which at the subgraphs involved in the move look like normal \mathcal{N} -graph moves).*

FIGURE 15. Shaeffer's b -transit.FIGURE 16. Branched realization of MP pb -transits.

Proof. The proof for bubble moves is easy and basically illustrated in Figure 14. Let us consider a positive MP \tilde{b} -transit $(T, \tilde{b}) \rightarrow (T', \tilde{b}')$. At the common 2-face F of the two tetrahedra of the initial sub-pattern of $(T, \sigma_{\tilde{b}})$, we can have either:

- (1) the two pre-branched tetrahedra share exactly two square edges which are both *not monochromatic* (i.e. each inherits *different* A, B labels from the two tetrahedra);
- (2) the two pre-branched tetrahedra share exactly one square edge which is monochromatic (either A or B), and the other two edges are monochromatic (either B or A coupled with the empty label).

Moreover, this information at F completely determines the pre-branching transit $(T, \sigma_{\tilde{b}}) \rightarrow (T', \sigma_{\tilde{b}'})$. Then it is enough to realize all the possibilities by means of b -transits. In Figure 16 we show that this is possible in terms of (decoded) normal \mathcal{N} -graphs. Clearly this realization is not unique. \square

6.2. Invariance. We consider weakly branched QH triangulations (T, \tilde{b}, w, f, c) encoding some pattern \mathcal{P} . Recall from Section 5 that QHFT patterns include a non empty link L , and we deal with distinguished triangulations (T, H) of (\hat{V}, L) where H is encoded by the global charge c . For patterns based on cusped manifolds, L and H are immaterial. In any case the decoration (w, f, c) verifies the necessary global constraints described in Section 3 and 5. If we just perform a change of weak branching $\tilde{b} \rightarrow \tilde{b}'$, the new QH triangulation $(T, \tilde{b}', w', f', c')$ is such that, for every (Δ, b) of (T, \tilde{b}) on which the change of weak branching $\tilde{b} \rightarrow \tilde{b}'$ is realized as $b \rightarrow b_\beta$ for some $\beta \in S(J_4)$, there is a determined $\delta(\beta) \in S(J_3)$ such that, for every $j = 0, 1, 2$,

$$w'_{\delta(\beta)(j)} = ((w_j)^{*b})^{*b'}, \quad f'_{\delta(\beta)(j)} = *_{b'}(*_b f_j), \quad c'_{\delta(\beta)(j)} = c_j.$$

Let us consider for a while genuinely branched triangulations (T, b) of \hat{V} , as in [1, 2, 3]. The main invariance property of the non-enhanced QH state sums $\mathcal{H}_N(T, b, c)$ (see (28) & (29)) is the invariance of their values under the *QH transits* of QH triangulations:

$$(T, b, w, f, c) \rightarrow (T', b', w', f', c') .$$

In particular, under such transits a “naked” triangulation move $T \rightarrow T'$ is enhanced to a *b-transit* $(T, b) \rightarrow (T', b')$. Moreover, (w, f, c) is equal to (w', f', c') on the “common” tetrahedra of T and T' , that is the ones which are not involved in the move. For the other tetrahedra, in the case of a QH transit based on a *MP* move $T \rightarrow T'$, the couples (w, w') form a *rational relation* $R(T \rightarrow T')$ in the product of gluing varieties $G(T) \times G(T')$. In coordinates (z, z') , this relation is defined by a finite set of equations of the form $z_j = z'_r z'_s$ or $z'_j = z_r z_s$. It is not hard to verify that two QH triangulations related by a QH transit encode a same pattern \mathcal{P} .

The proof [1, 2, 3] that

$$\mathcal{H}_N(\mathcal{P}) \equiv_{2N} \mathcal{H}_N(T, b, w, f, c)$$

is a well defined invariant of the pattern \mathcal{P} consists in proving that given two such triangulations (T, b, w, f, c) and (T', b', w', f', c') , there is a finite sequence

$$(68) \quad (T_0, b_0, w_0, f_0, c_0) \rightarrow (T_1, b_1, w_1, f_1, c_1) \rightarrow \cdots \rightarrow (T_k, b_k, w_k, f_k, c_k)$$

such that

- $(T_0, b_0, w_0, f_0, c_0) = (T, b, w, f, c)$
- $(T_k, b_k, w_k, f_k, c_k) = (T', b', w', f', c')$
- $(T_{j+1}, b_{j+1}, w_{j+1}, f_{j+1}, c_{j+1})$ is obtained from $(T_j, b_j, w_j, f_j, c_j)$ by performing a QH-transit, with the possible exception of the last step which can be a change of branchings $b_{k-1} \rightarrow b_k$.

Assuming this fact, *the full invariance is reduced to prove the invariance under a change of branching.*

Remark 6.4. It is known that two naked triangulations of \hat{V} with the same number $n \geq 2$ of tetrahedra can be connected by using only MP moves. However, there are negative $3 \rightarrow 2$ moves $T \rightarrow T'$ and branching b on T which cannot be enhanced to any *b-transit* $(T, b) \rightarrow (T', b')$. Similarly there are MP moves $T \rightarrow T'$ and points z in $G(T)$ such that there is no couples (z, z') in the rational relation $R(T \rightarrow T')$. It is necessary to use also the bubble moves to overcome this possible stop to enhancing the naked moves. As a consequence, the above description of the invariance proof is faithful only in the case of QHFT patterns, while in the the case of patterns based on cusped manifolds, even if the initial and final triangulations are ideal ones, we have to allow suitably structured non-ideal triangulations in the intermediate terms of a sequence like (68), keeping nevertheless the invariance of the values of the state sum (see [2, 3] for the details). The same considerations apply to the proof of Theorem 1.1.

Let now (T, \tilde{b}, w, f, c) be a *weakly* branched QH triangulation that encodes a pattern \mathcal{P} . We want to prove similarly as above that

$$\mathcal{H}_N^e(\mathcal{P}) \equiv_{4N} \mathcal{H}_N^e(T, \tilde{b}, w, f, c)$$

is a well defined invariant of the pattern \mathcal{P} . Let $(T', \tilde{b}', w', f', c')$ be another QH triangulation that encodes \mathcal{P} . By using Lemma 6.3 and the constructions of [1, 2, 3], it is not hard to show that there exists a sequence of QH triangulations $(T_j, \tilde{b}_j, w_j, f_j, c_j)$, $j = 0, \dots, k$, that connects (T, \tilde{b}, w, f, c) to $(T', \tilde{b}', w', f', c')$ as in (68), and such that $(T_{j+1}, \tilde{b}_{j+1}, w_{j+1}, f_{j+1}, c_{j+1})$ is obtained from $(T_j, \tilde{b}_j, w_j, f_j, c_j)$ by performing either a change of weak branching $\tilde{b}_j \rightarrow \tilde{b}_{j+1}$ (again we can assume that all of them are oriented *C-moves* except possibly the last one), or a QH transit that is locally supported by a genuine *b-transit*. As it is evident that the values of the enhanced QH state sums are also invariant under such a kind of QH transit, *their full invariance is also reduced to prove the invariance under a change of weak branching.* This is what we are going to prove in the rest of this Section.

6.2.1. *Formal conversion of tensor networks.* Recall the notation $V = \mathbb{C}^N$. We use the canonical isomorphism of $V \rightarrow V^*$ that sends the standard basis of V onto the dual basis of V^* .

Let $A \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ be associated to a $2-2$ \mathcal{N} -tangle as in Figure 3, according to the conventions fixed in Section 3.1. Let $\beta \in S(J_4)$. Under the natural isomorphisms

$$(69) \quad \text{Hom}(V_3 \otimes V_1, V_2 \otimes V_0) \cong V_2 \otimes V_0 \otimes (V_3 \otimes V_1)^*$$

$$(70) \quad \cong V_2 \otimes V_0 \otimes V_1^* \otimes V_3^*,$$

assuming that $*_b = 1$, we see that A belongs to $V_2 \otimes V_0 \otimes V_1^* \otimes V_3^*$, where as usual V_j is the copy of \mathbb{C}^N associated to the j th-face of (Δ, b) . Consider $W_2 \otimes W_0 \otimes W_1^* \otimes W_3^*$ if $*_{b_\beta} = 1$, and $W_3 \otimes W_1 \otimes W_0^* \otimes W_2^*$ if $*_{b_\beta} = -1$, where W_j is the copy of \mathbb{C}^N associated to the j th-face according to b_β . By using either $\text{Id} : V_j \rightarrow W_{\beta(j)}$ or the canonical isomorphisms $V_j \rightarrow W_{\beta(j)}^*$, we get further canonical isomorphisms

$$V_2 \otimes V_0 \otimes V_1^* \otimes V_3^* \rightarrow W_2 \otimes W_0 \otimes W_1^* \otimes W_3^*$$

and

$$V_2 \otimes V_0 \otimes V_1^* \otimes V_3^* \rightarrow W_1 \otimes W_3 \otimes W_0^* \otimes W_2^*.$$

Denote by A_β the operator supported by (Δ, b_β) , defined as the image of A via such an isomorphism. We call A_β the *formal β -conversion* of A . There are explicit identities between the matrix elements of A and A_β . Following our conventions for index positions, we get for example

$$\begin{aligned} A_{s,t}^{p,q} &= (A_{(13)(02)})_{t,s}^{q,p} = (A_{(23)})_{s,q}^{p,t} = \\ &= (A_{(02)})_{q,p}^{s,t} = (A_{(01)})_{p,t}^{s,q} = (A_{(12)})_{t,q}^{s,p}. \end{aligned}$$

If \mathcal{A} is, as for QH state sums, a network of tetrahedral tensors supported by a weakly branched triangulation (T, \tilde{b}) , for every weak branching transition $\tilde{b} \rightarrow \tilde{b}'$ the formal conversions of the vertex endomorphisms of \mathcal{A} match to produce a new tensor network $\mathcal{A}_{\tilde{b}'}$, supported by (T, \tilde{b}') , called the *\tilde{b}' -formal conversion* of \mathcal{A} . It is tautologically evident that the associated state sums have the same value:

$$\text{trace}(\mathcal{A}) = \text{trace}(\mathcal{A}_{\tilde{b}'}).$$

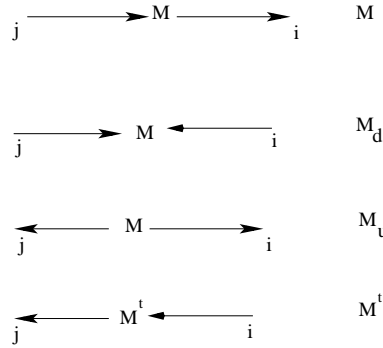


FIGURE 17. Formal conversion of a square matrix.

$$\begin{aligned} \xrightarrow{i} A \xleftarrow{k} B \xrightarrow{j} &= \xrightarrow{i} AB \xrightarrow{j} \\ \xleftarrow{i} A \xrightarrow{k} C \xleftarrow{j} &= \xleftarrow{i} C \xrightarrow{k} A \xleftarrow{j} \end{aligned}$$

FIGURE 18. Relations for M -conversion.

The formal conversion of tensors can be widely applied. Here is another example. Let M be any $N \times N$ matrix. We interpret $M = (M_j^i)$ as an endomorphism of $V = \mathbb{C}^N$. There are two formal

conversions of M , denoted M_u and M_d , which are bilinear forms on V^* and on V respectively, defined by the matrix element identities:

$$(71) \quad (M_u)^{i,j} = M_j^i = (M_d)_{i,j} .$$

Note that we have also the identities

$$(72) \quad (M_u)^{i,j} = (M^t)_i^j = (M_d)_{i,j} .$$

where M^t is the transpose matrix of M , that is the adjoint endomorphism. In Figure 17 we show a graphical encoding of this definition. Note that if M is symmetric, then the actual position of the indices i, j is immaterial. These conversions satisfy a few relations graphically shown in Figure 18, where A, B and C denote matrices and we assume that $AC = CA$. Note that in every case the index k is traced out, and the top/right picture represents the matrix element of a composite endomorphism $AB = (AB)_j^i$.

Graphical algorithm of the formal conversion:

- Take a $2 - 2$ \mathcal{N} -tangle as in Figure 3, that carries an endomorphism $A \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$. Introduce near each endpoint of each arc of the tangle an auxiliary internal vertex. Forget the orientations on the four ending intervals.
- Let $\beta \in J_4$. Label and orient the ending intervals in agreement with the second and third steps of the algorithm for the tangle $C(b_\beta)$ in Section 6.1.2.
- Every auxiliary internal vertex that we have just introduced is now a common endpoint of two oriented intervals. These orientations can be compatible or not. In any case, separate the two intervals and introduce the identity matrix I , according to Figure 17. We get a $2 - 2$ tangle which is also a tensor network. It is a graphical presentation of a formal conversion of A . By using our index conventions, the J_4 -labelling allows one to identify it with A_β .
- Let $\mathcal{A}_{\tilde{b}}$ be a tensor network supported by (T, \tilde{b}) , and $\mathcal{A}_{\tilde{b}'}$ the formal conversion of $\mathcal{A}_{\tilde{b}}$ associated to a weak branching transit $\tilde{b} \rightarrow \tilde{b}'$. Perform the previous graphical conversion at each vertex of a \mathcal{N} -graph that supports $\mathcal{A}_{\tilde{b}}$. The resulting tensor network coincides with $\mathcal{A}_{\tilde{b}'}$.

Here is an important remark to do:

Remark 6.5. Strictly speaking, in order to recover $\mathcal{A}_{\tilde{b}'}$ in the last step of the algorithm we need also the information carried by the J_4 -labelling. However, we are only interested in the trace of $\mathcal{A}_{\tilde{b}'}$, for which this information is immaterial, only the edge orientations being relevant. So we can forget the J_4 -labelling.

6.2.2. Relations between $\mathcal{R}_N(\Delta, b, \delta)$ and $\mathcal{R}_N(\Delta, b_\beta, \delta')$. Consider a QH tetrahedron (Δ, b, δ) , $\delta = (w, f, c)$. Let $\beta \in S(J_4)$. $(\Delta, b_\beta, \delta')$ is obtained from (Δ, b, δ) as stated at the beginning of Section 6.2. Our next task is to point out the relation between the non-enhanced matrix dilogarithms $\mathcal{R}_N(\Delta, b, \delta)$ and $\mathcal{R}(\Delta, b_\beta, \delta')$. At least in principle, it is enough to establish such relations when $\beta \in \{(01), (12), (23)\}$, as such transpositions generate the group $S(J_4)$. We will provide a graphical treatment, similarly to what we have done above for the formal conversion.

Recall the matrices S and T defined in Section 3.1.2. It is time to give a

Proof of Lemma 3.2. As usual we put $m = (N - 1)/2$, $\zeta = e^{2\pi\sqrt{-1}/N}$. We have

$$\begin{aligned} (S^2)_i^j &= N^{-1} \sum_{k=0}^{N-1} \zeta^{k(i+j)} = \delta_N(i+j) \\ (ST)_i^j &= N^{-1/2} \sum_{k=0}^{N-1} \zeta^{ik+(m+1)k^2} \delta_N(k+j) = N^{-1/2} \zeta^{-ij+(m+1)j^2} \\ ((ST)^2)_i^j &= N^{-1} \sum_{k=0}^{N-1} \zeta^{-k(i+j)+(m+1)(k^2+j^2)} \\ &= N^{-1} \zeta^{(m+1)(j^2-(i+j)^2)} \sum_{k=0}^{N-1} \zeta^{(m+1)(k-i-j)^2}. \end{aligned}$$

For all non vanishing coprime integers a, b with $b > 0$ set

$$G(a, b) = \sum_{x \bmod(b)} e^{2\pi\sqrt{-1}ax^2/b}.$$

The sum in $((ST)^2)_i^j$ has this form for $a = m + 1$, $b = N$. By [26, page 86-87] we have

$$G(a, b) = \left(\frac{a}{b}\right) G(1, b), \quad b \text{ odd}$$

where $\left(\frac{a}{b}\right) \in \{-1, 1\}$ is the Legendre-Jacobi quadratic symbol, and

$$G(1, b) = \begin{cases} \sqrt{b} & \text{if } b \equiv 1 \pmod{4} \\ i\sqrt{b} & \text{if } b \equiv 3 \pmod{4} \end{cases}$$

Hence

$$((ST)^2)_i^j = \begin{cases} \left(\frac{m+1}{N}\right) N^{-1/2} \zeta^{-(m+1)i^2-ij} & \text{if } N \equiv 1 \pmod{4} \\ \left(\frac{m+1}{N}\right) i N^{-1/2} \zeta^{-(m+1)i^2-ij} & \text{if } N \equiv 3 \pmod{4} \end{cases}$$

Finally

$$((ST)^3)_i^j = \mu_N N^{-1} \sum_{k=0}^{N-1} \zeta^{-ik+(m+1)k^2-(m+1)k^2-kj} = \phi_N \delta_N(i+j) = \phi_N S^2$$

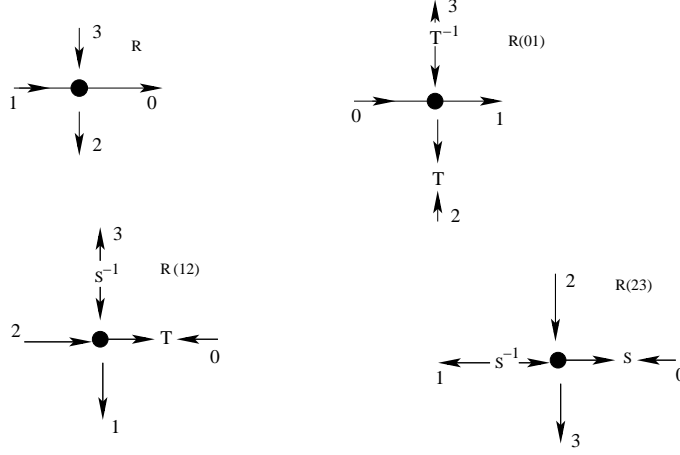
As $TS^{-1} = S(S^{-1}T)S^{-1}$, it is enough to prove the last statement for $S^{-1}T$. This follows: $((ST)^2)^3 = \phi_N^2 \mathbf{I}_N$; $((ST)^2)^2 = (ST)^3(ST) = \phi_N S^3 T = \phi_N S^{-1} T$; $(\phi_N S^{-1} T)^3 = ((ST)^4)^3 = \phi_N^4 \mathbf{I}_N$; finally $(S^{-1}T)^3 = \phi_N \mathbf{I}_N$. \square

Graphical algorithm to produce $\mathcal{R}(\Delta, b_\beta, \delta')$ when $\beta = (01), (12), (23)$:

- Apply to $A = \mathcal{R}(\Delta, b, \delta)$ the first three steps of the algorithm of formal conversion.
- J_4 -label the auxiliary internal vertices according to b , and give the i th vertex the element $\hat{e}_\beta(i) \in S(J_3)$ as in the last step of the algorithm for the tangle $C(b_\beta)$, in Section 6.1.2. The following Lemma is an easy verification.

Lemma 6.6. *For $\beta = (01), (12), (23) \in S(J_4)$ the following facts hold:*

- (1) *for every $i \in J_4$, then either $\hat{e}_\beta(i) = \text{Id}$ or $\hat{e}_\beta(i) \in \{(01), (12)\} \subset S(J_3)$.*
 - (2) *There are exactly two indices such that $\hat{e}_\beta(i) \neq \text{Id}$. They are associated to the two auxiliary internal vertices where the orientations of the incident arcs are conflicting.*
 - (3) *One of these vertices is a source, the other is a pit.*
- At every auxiliary vertex such that $\hat{e}_\beta(i) = \text{Id}$, keep the identity matrix I introduced in the formal conversion. If $\hat{e}_\beta(i) = (01)$, then replace I with T^h ; if $\hat{e}_\beta(i) = (12)$, then replace I with S^h , where $h = \pm 1$. Finally the sign h is determined by the following rule: $h = 1$ (resp. $h = -1$) if the vertex is a pit (resp. a source).

FIGURE 19. $\mathcal{R}_N(\Delta, b_\beta, \delta')$.

By tracing the so obtained tensor tangle (with J_4 -labelled free ends) we get a tensor of the same type as $\mathcal{R}(\Delta, b_\beta, \delta')$. In fact the content of Lemma 3.3 of [1] and Proposition 5.3 of [2] can be rephrased as follows.

Proposition 6.7. *If $\beta = (01), (12), (23)$, then the tensor obtained from $\mathcal{R}_N(\Delta, b, \delta)$ by applying the above algorithm coincides with $\mathcal{R}(\Delta, b_\beta, \delta')$, up to the usual phase ambiguity (\equiv_{2N}). In formulas:*

$$\mathcal{R}_N(\Delta, b_{(01)}, \delta') \equiv_{2N} T_1^{-1} \mathcal{R}_N(\Delta, b, \delta) T_1$$

$$\mathcal{R}_N(\Delta, b_{(12)}, \delta') \equiv_{2N} S_1^{-1} \mathcal{R}_N(\Delta, b, \delta) T_2$$

$$\mathcal{R}_N(\Delta, b_{(23)}, \delta') \equiv_{2N} S_2^{-1} \mathcal{R}_N(\Delta, b, \delta) S_2 .$$

In Figure 19 we show the graphical representation of these formulas, assuming for simplicity that $*_b = 1$.

Remark 6.8. (On the computation of $\mathcal{R}(\Delta, b_\beta, \delta')$ for arbitrary $\beta \in S(J_4)$.) Let us write β as a reduced word, say

$$\beta = \tau_k \circ \dots \circ \tau_1$$

in the alphabet of generators $(01), (12), (23)$. For every $i \in J_4$, let $\hat{e}_\beta(i) \in S(J_3)$ be as at the end of the algorithm for the tangle $C(b_\beta)$. Let us apply k times Proposition 6.7, by using $\tau_1, \tau_2, \dots, \tau_k$ consecutively. Forget the auxiliary vertices introduced time by time at which the incident arc orientations do agree (and at which we would insert the identity matrix). On every edge e_i (J_4 -labelled according to the branching b) of the initial tangle for $\mathcal{R}_N(\Delta, b, \delta)$, we are left with a sequence of auxiliary vertices where pits and sources alternate. We have

$$\hat{e}_\beta(i) = (\epsilon_{\tau_k}(i) \circ \dots \circ \epsilon_{\tau_1}(i))^e$$

for a suitable exponent $e = \pm 1$. As every $\epsilon_{\tau_j}(i) \in \{\text{Id}, (01), (12)\} \subset S(J_3)$, we have actually obtained a word $\omega_\beta(i)$ in the alphabet of generators $\{(01), (12)\}$, such that $\omega_\beta(i)^e$ (as an element of $S(J_3)$) coincides with $\hat{e}_\beta(i)$. In fact the accidental $\epsilon_{\tau_j}(i) = \text{Id}$ are immaterial. This word is not necessarily reduced, but we can consider the associated reduced word $\omega_\beta^r(i)$, by eliminating pairs of adjacent equal letters (if any). Let us look now at the resulting tensor tangle. Up to the usual phase ambiguity (\equiv_{2N}), it coincides with $\mathcal{R}_N(\Delta, b_\beta, \delta')$. Along every edge e_i , in correspondence with the word $\omega_\beta(i)$ we find a word in the alphabet of matrices $\{T^{\pm 1}, S^{\pm 1}\}$, where the exponents alternate. In correspondence with a pair of identical adjacent letters in $\omega_\beta(i)$, we get a term of the form MM^{-1} which can be cancelled. Hence, in correspondence with the reduced word $\omega_\beta^r(i)$, we get a reduced word, say $\Omega_\beta(i)$, in the alphabet $\{T^{\pm 1}, S^{\pm 1}\}$ where both the symbols S, T and the exponents alternate.

6.2.3. Invariance of the enhanced QH state sums under oriented C-moves. Let us apply the previous discussion to a permutation $\beta \in S(J_4)$ that preserves the pre-branching. Set $\tau = (02)(13)$, $\sigma = (0, 1, 2, 3) \in S(J_4)$. Then

$$\tau^{-1} = \tau, \sigma^{-1} = (0321), \sigma\tau = \tau\sigma = \sigma^{-1}$$

hence τ, σ generate a subgroup, say \mathfrak{S} , of $S(J_4)$ isomorphic to $(\mathbb{Z}/4\mathbb{Z}, +)$, via the isomorphism sending $\tau \rightarrow 2, \sigma \rightarrow 1$. Looking at Figure 2 anti-clockwise from the top/left, and denoting by b the first branching, we see that the other ones are in the order $b_\tau, b_{\sigma\tau}, b_{\tau\sigma\tau}, b_{\sigma^{-1}\tau\sigma\tau} = b$. Hence $\beta \in \mathfrak{S}$.

As β preserves the pre-branching, according to Figure 18 every word $\Omega(i)$ represents an endomorphism of the copy V_i of \mathbb{C}^N associated to e_i . Note that under the usual isomorphism $\mathbb{Z}/3\mathbb{Z} \rightarrow A(J_3)$, $j \rightarrow (012)^j$, for every $i \in J_4$ the permutation $\hat{e}_\beta(i)$ corresponds to the variation, say $t_i \in \mathbb{Z}/3\mathbb{Z}$, of the $\mathbb{Z}/3\mathbb{Z}$ \mathcal{N} -graph color $r(e_i)$ after the oriented C-move encoded by β (see Figure 12). Recall from Lemma 3.2 that

$$\mathcal{Q} := TS^{-1}$$

is (projectively) of order 3, up to the scalar factor ϕ_N which is a 4th root of unity. According to Remark 6.8 applied to $\beta \in \mathfrak{S}$, a priori the reduced words $\Omega_\beta(i)$ in the alphabet $\{T^{\pm 1}, S^{\pm 1}\}$ could involve powers of $\mathcal{Q} = TS^{-1}$ with exponents greater than 1 or lower than -1 . In order to reduce further $\Omega_\beta(i)$ to get exponents in $\{-1, 0, 1\}$, one may use the relation $(TS^{-1})^3 = \mu_N I_N$, but it would introduce a projective ambiguity by $4N$ th root of unity factors. Nevertheless, in the actual situation of $\beta \in \mathfrak{S}$, we easily realize that such a further reduction of the words $\Omega_\beta(i)$ is not necessary. Hence:

Proposition 6.9. *Let $\beta \in \mathfrak{S}$, and $t_i \in \{-1, 0, 1\}$ (identified with $\mathbb{Z}/3\mathbb{Z}$) the variation of the \mathcal{N} -graph color $r(e_i)$ after the oriented C-move encoded by β . Then for every $i \in J_4$ we have $\Omega_\beta(i) \equiv_{2N} \pm \mathcal{Q}^{t_i}$.*

As a corollary we obtain the invariance of the values of the enhanced state sums under the moves that preserve the pre-branching, and up to the better phase ambiguity:

Corollary 6.10. *Let (T, \tilde{b}, δ) and (T, \tilde{b}', δ') be QH weakly branched triangulations of a same pattern \mathcal{P} , that are just related by the change of weak branching $\tilde{b} \rightarrow \tilde{b}'$. Assume that \tilde{b} and \tilde{b}' induce the same pre-branching. Then*

$$\mathcal{H}_N^e(T, \tilde{b}, \delta) \equiv_{2N} \mathcal{H}_N^e(T, \tilde{b}', \delta') .$$

Proof. This is almost immediately a consequence of Proposition 6.9, as such a change of weak branching can be obtained by means of a sequence of oriented C-moves. The only point that deserves a comment is the following. Proposition 6.7, that we apply several times, includes informations about the J_4 -labellings, which apparently disappear in the discussion that leads to Proposition 6.9 and hence to the present statement. Considerations similar to the ones of Remark 6.5 explain why such J_4 -labellings are unessential: the tensor network whose trace is $\mathcal{H}_N^e(T, \tilde{b}', \delta')$ is a formal conversion of the network supported by (T, \tilde{b}, δ) . The invariance result is eventually proved. \square

6.2.4. Full invariance. We are ready to state our final invariance results.

Proposition 6.11. *Let (T, \tilde{b}, δ) and (T, \tilde{b}', δ') be QH weakly branched triangulation of a same pattern \mathcal{P} , that are just related by the change of weak branching $\tilde{b} \rightarrow \tilde{b}'$. Then*

$$\mathcal{H}_N^e(T, \tilde{b}, \delta) \equiv_{4N} \mathcal{H}_N^e(T, \tilde{b}', \delta') .$$

Proof. Let (Γ, θ) and (Γ', θ') be \mathcal{N} -graphs representing the two triangulations respectively. Let e be an edge of (Γ, θ) and \bar{e} the corresponding edge in (Γ', θ') . If e and \bar{e} have compatible orientations, we can apply the same arguments as for the oriented C-moves (Corollary 6.10). Assume that e and \bar{e} have opposite orientations. Recall the relation between the respective r -colors:

$$\hat{e}^f(\bar{e}) \circ (012)^{-r(e)} \circ \hat{e}^i(\bar{e}) = (012)^{r(\bar{e})}$$

where, in the present situation, both $\hat{e}^f(\bar{e})$ and $\hat{e}^i(\bar{e})$ are transpositions of $S(J_3)$. Assume for simplicity that $r(e) = 0$. If $\hat{e}^f(\bar{e}) = \hat{e}^i(\bar{e})$, then also $r(\bar{e}) = 0$ and the edge tensor is the identity, hence immaterial. On the other hand, at the corresponding pit/source of the edge e we get a suitable matrix M and its inverse respectively. (Note that in the case of genuine branchings only this case happens, so that we have recovered the proof of Lemma 4.15. of [1]). Assume that $\hat{e}^f(\bar{e}) \neq \hat{e}^i(\bar{e})$, and, again for simplicity,

that these are among the favourite generators (01), (12). Then at the pit/source of e we get either T , S^{-1} or S , T^{-1} . In the first case $r(\bar{e}) = 1$, hence the edge tensor at \bar{e} is $Q^{-1} = ST^{-1}$. In the second case $r(\bar{e}) = -1$, hence the edge tensor at \bar{e} is $Q = TS^{-1}$. By converting in both cases the resulting network of tensors in one endomorphism supported by e (by applying the relations of Figures 17 & 18), we realize that it is the identity (i.e. Q^0), like on the original edge e . The general case (that is for an arbitrary value of $r(e)$, and for the transpositions $\hat{e}^f(\bar{e}) \neq \hat{e}^i(\bar{e})$ possibly equal to (02)), follows by very similar considerations. \square

Remark 6.12. There is a slight difference between the invariance proofs given in [1] and [2]. In [1] we did as we described above. In [2], by using a result of Costantino saying that any change of branching is realized by b -transits [11], we achieved the proof by dealing only with QH transits. This suggests the following

Question 6.13. *Let (T, \tilde{b}) and (T, \tilde{b}') be weakly-branched triangulations of \hat{V} that differ only by weak branchings. Are they \tilde{b} -equivalent ?*

A positive answer would imply that the ambiguity “ \equiv_{4N} ” is too prudential, since then we could deal only with oriented C -moves, apply Corollary 6.10, and eventually get that also the enhanced QH invariants would be defined up the ambiguity “ \equiv_{2N} ”.

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